Probability Qualifying Exam

January 2023

Instructions (Read before you begin)
- You may attempt all 6 problems in this exam. However, you can turn in solutions for at most 4 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is 10 points; a pass is 30 points or higher; a high pass is 36 points or higher.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Without using the Central Limit Theorem or the Law of the Iterated Logarithm, prove the Weak Law of Large Numbers when \( X_i \in L^1 \), i.e. show that when \( X_1, X_2, \ldots \) is a sequence of i.i.d. variables with \( \mathbb{E}[X_1] = \mu \) and \( X_1 \in L^1 \), then

\[
\bar{X} = \frac{X_1 + \ldots + X_n}{n} \to \mu \quad \text{in} \quad L^1.
\]

Hint: first, prove the claim when \( X_1 \in L^2 \). Then for \( a > 0 \), consider \( X_i^a = X_i \cdot 1\{|X_i| \leq a\} \).

2. Let \( \{A_i\}_{i=1}^\infty \) be a sequence of independent events such that \( \sum_{i=1}^\infty \mathbb{P}(A_i) = \infty \). Prove that the \( A_n \) occur infinitely often with probability 1 (the converse of the Borel-Cantelli Lemma), i.e. that \( \sum_{i=1}^\infty 1_{A_i} = \infty \) almost surely.

3. Show that if \( X_n \to X \) in probability, then \( X_n \Rightarrow X \), i.e. \( X_n \) converges weakly to \( X \).

4. Recall the inversion formula: if \( \mu \) is a probability measure and \( \hat{\mu} \) is its characteristic function, then for all \( a < b \):

\[
\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \int_a^b e^{-ity} \hat{\mu}(t) \, dy \, dt = \mu(a, b) + \frac{1}{2} \mu(\{a, b\}).
\]
(a) Prove that if $\hat{\mu} \in L^1$, then $\mu(\{a\}) = 0$ for all $a \in \mathbb{R}$ and $\mu$ has a probability density function given by

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \hat{\mu}(t) \, dt.$$ 

(b) Show that if $X_1, \ldots, X_n$ are independent and uniformly distributed on $(-1, 1)$, then for $n \geq 2$, $X_1 + \ldots + X_n$ has density

$$f(y) = \frac{1}{\pi} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^n \cos(ty) \, dt.$$ 

5. Prove that if $\{X_i\}_{i=1}^{\infty}$ are i.i.d. Uniform-$[0,1]$ random variables, then

$$\frac{4 \sum_{i=1}^{n} i X_i - n^2}{n^2}$$

converges weakly, and identify the limiting distribution. (Recall $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.)

6. Let $X_n, Y_n$ be positive, in $L^1$, and measurable with respect to the filtration $\mathcal{F}_n$. Suppose

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] \leq (1 + Y_n) X_n$$

with $\sum Y_n < \infty$ a.s.

(a) Show that $\prod_{i=1}^{n} (1 + Y_i)$ converges a.s. to a finite limit.

(b) Show

$$M_n = \frac{X_n}{\prod_{i=6}^{n-1} (1 + Y_i)}$$

is a super-martingale.

(c) Use (a) and (b) to prove that $X_n$ converges a.s. to a finite limit.