UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. preliminary Examination on
Applied Linear Operators and Spectral Methods (Math 6710)
August 20, 2020

Instructions: This examination includes five problems but you are to work three of them. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three problems will be graded. All problems are worth 20 points. A pass is 35 or more points. A high-pass is 45 or more points.

1. Let $T : X \to Y$ be a compact linear operator defined on infinite dimensional Banach spaces X and Y. Let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$.
   (a) Show that the closed unit ball $B = \{x \in \mathcal{N}(T - \lambda I) \mid \|x\| \leq 1\}$ is compact.
   (b) Does the previous result say anything about $\dim \mathcal{N}(T - \lambda I)$?
2. Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Show that T is compact if and only if T^*T is compact.
3. Let H be an infinite dimensional Hilbert space and let $(e_n) \subset H$ be a total orthonormal sequence. Let $x \in H$ with $\|x\| \leq 1$. Construct a sequence $(x_n) \subset H$ such that $\|x_n\| = 1$ and $x_n \to x$ **weakly**.
4. Let H be a separable Hilbert space with (e_n) being a total orthonormal sequence of H. Let $T : H \to H$ be a bounded linear operator satisfying

$$\sum_{n=1}^{\infty} \|Te_n\| < \infty.$$

   Show that T is compact by approximating T by a sequence (T_k) of finite rank operators that converges to T in an appropriate sense.

(continued in next page)

5. The goal of this problem is to show that for a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ we have $\partial_n u = 0$ if and only if we can find a distribution $v \in \mathscr{D}'(\mathbb{R}^{n-1})$ such that

$$\langle u, \phi \rangle = \left\langle v(x'), \int \phi(x', x_n)\, dx_n \right\rangle, \quad \text{for any } \phi \in C_c^\infty(\mathbb{R}^n). \tag{1}$$

Here we write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$.

(a) Show that if $v \in \mathscr{D}'(\mathbb{R}^{n-1})$ then the distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ defined by (1) is such that $\partial_n u = 0$.

(b) From now on let $u \in \mathscr{D}'(\mathbb{R}^n)$ be such that $\partial_n u = 0$ and choose some $\chi \in C_c^\infty(\mathbb{R})$ such that

$$\int \chi(t) dt = 1.$$

Define $v \in \mathscr{D}'(\mathbb{R}^{n-1})$ by

$$\langle v, \psi \rangle = \langle u, \psi(x')\chi(x_n) \rangle, \quad \text{for any } \psi \in C_c^\infty(\mathbb{R}^{n-1}).$$

For some $\phi \in C_c^\infty(\mathbb{R}^n)$, find a function $\widetilde{\phi} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\langle u, \phi \rangle - \left\langle v(x'), \int \phi(x', t)\, dt \right\rangle = \left\langle u, \widetilde{\phi} \right\rangle.$$

(c) Show that $\left\langle \partial_n u, \int_{x_n}^\infty \widetilde{\phi}(x', t)\, dt \right\rangle = \left\langle u, \widetilde{\phi} \right\rangle$. You may assume that $\widetilde{\phi} \in C_c^\infty(\mathbb{R}^n)$ and that

$$\int_{x_n}^\infty \widetilde{\phi}(x', t)\, dt \in C_c^\infty(\mathbb{R}^{n-1}).$$

(d) Use the hypothesis $\partial_n u = 0$ to conclude.