Preliminary Exam, Numerical Methods for ODEs and PDEs
August 2020

Instructions: This exam is closed book, no notes, and no electronic devices are allowed. You have two hours and you need to work on any 3 out of the 5 questions. All questions have equal weight and a score of 65% is considered a pass, and a score of 80% is considered a high pass. Indicate clearly the work that you wish to be graded.

Note: In the following problems, the notation $k = \Delta t$ and $h = \Delta x$ is used.

<u>**Problem 1.**</u> (Elliptic Problems)

Consider the one dimensional problem for $v(x)$

$$v''(x) = f(x), \tag{1}$$

in the interval [0,1] along with homogeneous Dirichlet boundary conditions. Define the difference operator

$$\Delta_h U_j^h \equiv \frac{1}{h^2}\Big(U_{j-1}^h - 2\;U_j^h + U_{j+1}^h\Big),$$

and consider the scheme for Eq. 1

$$\Delta_h U_j^h = f_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$, $f_j \equiv f(jh)$, and $U_0^h = U_N^h = 0$. Here, the superscript h is used to indicate the grid size, and it is hoped that $U_j^h \approx v(x_j)$. The approximate solution satisfies a linear system $AU^h = b$, where $U^h = (U_1^h, U_2^h, ..., U_{N-1}^h)^T$ and $b = h^2(f_1, f_2, ..., f_{N-1})^T$.

(a) State and prove the maximum principle for any grid function $V^h = \{V_j^h\}$ with values for $j = 0, 1, ...N$, that satisfies $\Delta_h V_j^h \geq 0$ for $j = 1, 2..., N-1$. Sketch a grid function for which $\Delta_h V_j^h \geq 0$.
(b) Derive an expression for the local truncation error and find the equation that relates the local trunction error and the global error $e_j^h = v(x_j) - U_j^h$.
(c) Use the maximum principle from part (a) to show that $\|e^h\|_\infty = O(h^2)$ as the space step $h \to 0$.
(d) Prove that the matrix A is nonsingular.

Problem 2. (Heat Equation Stability)

Consider the variable coefficient diffusion equation

$$v_t = (\beta(x) v_x)_x, \qquad 0 < x < 1,\ t > 0$$

with Dirichlet boundary conditions

$$v(0,t) \;=\; 0, \qquad v(1,t) \;=\; 0$$

and initial data $v(x,0) = f(x)$. Assume that $\beta(x) \geq \beta_0 > 0$, and that $\beta(x)$ is smooth. Let $\beta_{j+1/2} = \beta(x_{j+1/2})$. A finite-difference scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{h^2} \quad \left\{ \beta_{j-1/2} u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2}) u_j^{n+1} + \beta_{j+1/2} u_{j+1}^{n+1} \right\}.$$

Here, u_j^n is meant to approximate $v(x_j, t_n)$ where $x_j = jh$ and $t_n = nk$. Analyze the 2-norm stability of this scheme for solving the above initial boundary value problem.

DO NOT NEGLECT THE FACT THAT THE PROBLEM HAS VARIABLE COEFFICIENTS AND THAT THERE ARE BOUNDARY CONDITIONS AT 0 AND 1!

<u>Problem 3.</u> (Numerical Methods for ODE Initial Value Problems)

Consider the Linear Multistep Method

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}kf_{n+2}$$

for solving an initial value problem $y' = f(y, x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y uniformly for all x.

(a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

(b) Sketch a graph of the solution to the following initial value problem.

$$y' = -10^8[y - \cos(x)] - \sin(x), \quad y(0) = 2.$$

Would it be more reasonable to use this method or the forward Euler method for this problem? What issues should be considered in choosing a timestep k for each of the methods? Justify your answer.

Problem 4. (Higher Order Methods)

Consider the following problem for $v(x)$ on [0,1]:

$$v''(x) = f(x),$$

with $v(0) = v(1) = 0$. Let $N \cdot h = 1$ and define $x_j = j \cdot h$ for $j = 0, 1, ..., N$. The finite-difference scheme

$$\Delta_h U_j^h \equiv \frac{1}{h^2}\left(U_{j-1}^h - 2U_j^h + U_{j+1}^h\right) = f_j^h,$$

for $j = 1, ..., N-1$ with $U_0^h = U_N^h = 0$ gives values U_j^h that approximate $v(x_j)$ with an error of $O(h^2)$. Here the superscript h is used to indicate the grid size for the solution. Show how to use this method to find a numerical solution W_j^h whose values approximate $v(x_j)$ with an error of $O(h^4)$ and a numerical solution Y_j^h whose values approximate $v(x_j)$ with an error of $O(h^6)$.

Problem 5. (Advection Equation)
Consider the equation $v_t + cv_x = 0$ for $-\infty < x < \infty$, and $t > 0$. Suppose the initial data is as smooth as you want and has compact support (that is, it is 0 outside a closed bounded interval). Consider the Lax-Wendroff scheme for finding an approximate numerical solution to this problem:

$$\frac{u_j^{n+1} - u_j^n}{k} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{1}{2}c^2 k \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}.$$

(a) Derive the equation satisfied by the global error $e_j^n = v(x_j, t_n) - u_j^n$ of the numerical solution obtained with this scheme.

(b) Show directly by analyzing the 2-norm of the global error that this scheme converges for this problem provided $|\alpha| \leq 1$ where $\alpha = \frac{ck}{h}$. It is not enough to just quote a theorem here.

(c) Give an explanation of how, even without this analysis, we know that the scheme does **not** converge if $|\alpha| > 1$.