Statistics Qualifying Examination

January 7, 2009

There are 10 problems, of which you should turn in solutions for **exactly** 6 (your best 6, in your opinion). Each problem is worth 10 points, and 40 points is required for passing. On the outside of your exam book, indicate which 6 you have attempted.

If you think a problem is misstated, interpret it in such a way as to make it nontrivial.

1. Let X_1 and X_2 be independent, each with probability density function $f(x) = 1/x^2$, $x > 1$. Find the joint probability density function of $Y_1 = X_1X_2$ and $Y_2 = X_1/X_2$. It is important to state where your formula is valid. (If you have time, check that your joint pdf integrates to 1; if it doesn't, find the error.)

2. Let $X_1, \ldots, X_n$ be independent LOGN(μ, σ^2) (i.e., $X_1 = e^{Y_1}, \ldots, X_n = e^{Y_n}$ with $Y_1, \ldots, Y_n$ being independent $N(\mu, \sigma^2)$). Argue that the sample median is asymptotically normal and find the asymptotic mean and variance.

3. Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, \theta)$, where $\theta > 0$. In particular, the population mean and the population variance are equal but unknown.

   (a) Find the method-of-moments estimator of θ based on the *second* sample moment.

   (b) Find the maximum likelihood estimator of θ.

4. Let $X_1, \ldots, X_n$ be a random sample from a $N(\theta, 1)$ distribution, and let the prior distribution of θ be $N(\mu_0, 1)$, where μ_0 is known.

   (a) Find the posterior distribution of θ.

   (b) Find the Bayes estimator of θ under squared error loss, and show that it is a weighted average of the prior mean μ_0 and the sample mean $\overline{X}$.

5. Let $X_1, \ldots, X_n$ be a random sample from Poisson(θ).

   (a) Find a complete, sufficient statistic.

   (b) Noting that $\mathrm{P}_\theta(X_1 = 1) = \theta e^{-\theta}$, use the Rao–Blackwell and Lehmann–Scheffé theorems to find a UMVUE of $\tau(\theta) := \theta e^{-\theta}$.

6. Assume a sample of size n from UNIF$[\eta - \theta, \eta + \theta]$, which is a location-scale family. Here η is real and $\theta > 0$.

   (a) Find a $100(1-\alpha)\%$ confidence interval for θ.

   (b) Find a $100(1-\alpha)\%$ confidence interval for η.

   It is not necessary to find the explicit distribution of the pivotal quantities; just denote the needed quantiles by x_p and y_p for an appropriate p.

7. Let $X_1, \ldots, X_n$ be a random sample from UNIF$[0, \theta]$. Find the UMP test of size α of $H_0 : \theta \geq \theta_0$ vs. $H_a : \theta < \theta_0$ by first deriving a most powerful test of simple hypotheses and then extending it to composite hypotheses.

8. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma_0^2)$, where σ_0^2 is known. Find the generalized likelihood ratio test of size α of $H_0 : \mu \geq \mu_0$ vs. $H_a : \mu < \mu_0$.

9. According to a genetic model the proportions of individuals having the four blood types should be given by O: q^2; A: $p^2 + 2pq$; B: $r^2 + 2qr$; AB: $2pr$. Here $p > 0$, $q > 0$, $r > 0$, and $p + q + r = 1$, but the three parameters are unknown. We observe n_O, n_A, n_B, n_{AB} individuals of the four blood types in a random sample of size n. Describe a goodness-of-fit test of size α of the stated hypothesis.

10. Consider the simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ $(i = 1, 2, \ldots, n)$. Assume independent $N(0, \sigma^2)$ errors.

    (a) Find the least squares estimators $\hat{\beta}_0$ of $\hat{\beta}_1$ of β_0 and β_1.

    (b) Calculate the covariance matrix of $(\hat{\beta}_0, \hat{\beta}_1)$.