DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do eight problems with at least three (3) problems from section A and three (3) problems from section B. Three completely correct problems in section A and two completely correct problems in section B is a passing exam. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Let μ be the outer measure on $\mathbb{R}$ defined by
$$\mu(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \mid A \subseteq \cup_{i=1}^{\infty}(a_i, b_i],\, a_i \leq b_i\}$$
for every $A \subset \mathbb{R}$. Prove that $\mu((a, b]) = b - a$. You may use that the length of a segment covered by finitely many segments is less than the sum of lengths of segments in the covering.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume there exists $\delta > 0$ such that for every $E \in \mathcal{M}$, either $\mu(E) = 0$ or $\mu(E) > \delta$. Let $f \in L^1(X)$. Prove that f is essentially bounded.

3. Let X be a normed space, and let X^* be the space of bounded linear functionals on X. For every $x \in X$, let x^* be the functional on X^* defined by $x^*(y) = y(x)$ for all $y \in X^*$. Prove that $x \mapsto x^*$ is an isometry from X into X^{**}.

4. Let V be a normed space and U a closed subspace. Define the norm on V/U and prove that V/U is complete if V is.

5. Let V be a Banach space, and $T : V \to V$ a bounded linear map. Assume that for every $v \in V$ there exists a non-negative integer n such that $T^n(v) = 0$. Prove that there exists an integer n such that $T^n(v) = 0$ for all $v \in V$.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

Notation: Let $\mathbb{H}$ be the upper half plane and $\mathbb{D}$ be the (open) unit disk.

6. Using the methods of complex analysis, compute $\int_0^\infty \frac{x^2}{x^4+1}dx$ and simplify your answer.

7. Show that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic then there exists $n_1, ...$ so that $f^{n_i}(z)$ converges pointwise for all $z \in \mathbb{D}$. Note that f^{n_i} denotes $\underbrace{f \circ ... \circ f}_{n_i \text{ times}}$.

8. Is there an entire function so that $f(\frac{1}{p}) = \frac{1}{1+p}$ for all primes p?

9. Show that if $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $f'(0) \neq 0$ then there is a neighborhood of 0 where f is injective. Do not use the inverse function theorem.

10. State and prove Schwartz's lemma.