DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS

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Instructions: You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded.

Real Analysis

Let λ denote Lebesgue measure, $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}$ and $\mathcal{H}$ denote Hilbert space.

1. Prove the Baire category theorem. That is, show that if (X, d) is a complete metric space and $U_1, ... \subset X$ are open and dense then $\cap_{i=1}^{\infty} U_i \neq \emptyset$.

2. (a) If $f_i \to g$ are in $L^1(\mathbb{R}, \lambda)$ and $\phi \in C(\mathbb{R})$ does $\lim\limits_{i\to\infty} \int f_i \phi d\lambda = \int g\phi d\lambda$?

   (b) Show that $f_i \to g$ in $L^1(\lambda, [0, 1])$ then there exists a subsequence $n_1, ...$ so that $f_{n_i} \to g$ almost everywhere.

3. If $v_i \in \mathcal{H}$ converges weakly to v_∞ and $\|v_\infty\| = \lim\limits_{i\to\infty} \|v_i\|$ then does v_i converge in norm to v_∞?

4. Let f is in $L^1(\lambda, \mathbb{R})$. Show that for any $r > 0$ the function $h(s) = \int_{s-r}^{s+r} f d\lambda$ is continuous.

5. (a) Let $A \subset [0, 1]$ and $A \in \mathcal{B}$. Show that for all $\epsilon > 0$ there exists a finite number of intervals $I_1, ..., I_n$ so that $\lambda(A\Delta(\cup_{j=1}^{n} I_j)) < \epsilon$.
   Note: Δ denotes symmetric difference. So $A\Delta B = \{x \in A \cup B : x \notin A \cap B\}$.

   (b) Show that we can not assume that $A \subset \cup_{j=1}^{n} I_j$.

Complex Analysis

Let $\mathbb{D}$ denote the (open) unit disk in $\mathbb{C}$. A region is an open connected subset of $\mathbb{C}$.

6. Prove or give a counter example: If Ω is a region, $f : \Omega \to \mathbb{C}$ admits unrestricted analytic continuation on a region $D \supset \Omega$ then there exists $F : D \to \mathbb{C}$, holomorphic and with $F|_\Omega = f$.

   Recall: If $f : \Omega \to \mathbb{C}$ is an analytic function we say f admits unrestricted analytic continuation on D if for any $\gamma : [0,1] \to D$ a piecewise smooth path, there exists an analytic continuation of (f, Ω) along the curve γ.

7. Classify entire biholomorphisms. That is, what are all $f : \mathbb{C} \to \mathbb{C}$ holomorphic with a holomorphic inverse $g : \mathbb{C} \to \mathbb{C}$?

8. Does there exists $f : \mathbb{D} \to \mathbb{C}$ holomorphic so that $f(\frac{1}{n})f(\frac{1}{n+1}) = \frac{1}{n}$ for all $n \in \{2, 3, ...\}$?

9. Let $\mathcal{F}$ be a family of holomorphic function on $\mathbb{D}$ so that there exists C with $|f(z)| < C$ for all $z \in \mathbb{D}$.

   (a) If $D \subset \mathbb{D}$ satisfies that $\bar{D} \subset \mathbb{D}$ then $\mathcal{F}$ is uniformly Lipshitz on D. That is, there exists u so that for any $f \in \mathcal{F}$ and $w, z \in D$ we have $|f(z) - f(w)| < u|z - w|$.

   (b) Show that $\mathcal{F}$ is not necessarily a uniformly Lipshitz family on $\mathbb{D}$.

10. What is $\int_{-\infty}^{\infty} \frac{(\cos(x))^2}{x^2+5} dx$?