DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS

August 2014

Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

Let λ denote Lebesgue measure.

1. Consider $L^2(\mathbb{Z}) := H$. Recall two topologies on the set of bounded linear operators on H: The norm topology which is metric topology given by $d(A, B) = \sup_{v \in H : ||v||=1} ||Av - Bv||$. The strong operator topology which is given by $A_i \to A_\infty$ if for every $v \in H$ we have $\lim_{i\to\infty} ||A_i v - A_\infty v|| = 0$.
   (a) Show that if A_i converges to A_∞ in the norm topology then it converges to A_∞ in the strong operator topology.
   (b) Give an example of A_i which converges to A_∞ in the strong operator topology but not the norm topology.
   (c) Let A_i be the operator defined by $A_i v = w$ where $w_j = v_j$ if $j \neq i$ and $w_i = iv_i$. Show $A_1, \ldots$ is a sequence of operators that does not converge in either topology.

2. Let $(X, \mathcal{B}, \mu)$ be a measure space. $U \in \mathcal{B}$ is called an *atom* if $\mu(U) > 0$ and $\mu(V) = 0$ for all V, a proper subset of U. If $(X, \mathcal{B}, \mu)$ is σ-finite measure space and $f : X \to Y$ then $f_*(\mu)$ has at most countably many atoms.

3. Let $f_i : [0, 1] \to \mathbb{R}$ be a sequence of functions that converge pointwise to f_∞.
   (a) Show that if the f_i are uniformly bounded then $\lim_{i\to\infty} \int f_i d\lambda = \int f_\infty d\lambda$.
   (b) What if $\|f_i\|_1 \leq 1$ for all i?
   (c) What if $\|f_i\|_2 \leq 1$ for all i?

4. Let $f \in L^1(\lambda, [0, 1])$ have the property that for any measurable set A with $\lambda(A) = \frac{1}{\pi}$ we have $\int_A f d\lambda = 0$. Show that $f = 0$ almost everywhere.

5. Let $A \subset \mathbb{R}$ and $A + A$ denote $\{a_1 + a_2 : a_1, a_2 \in A\}$. Show that if $\lambda(A^c) = 0$ then $A + A = \mathbb{R}$.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

6. Let f be a nonzero entire function. Assume that the set $Z(f) = \{z \in \mathbb{C} : f(z) = 0\}$ is unbounded. Show that f had an essential singularity at ∞.

7. Let f be an a holomorphic function defined on an open set $U \subset \mathbb{C}$ and $a \in U$. Assume that $f'(a)$ is nonzero. Show that for sufficiently small circles C centered at a
$$\frac{2\pi i}{f'(a)} = \int_C \frac{dz}{f(z) - f(a)}$$

8. Let f be an entire function and n a positive integer. Show that there exists an entire function g such that $f = g^n$ if and only if the order of each zero of f is divisible by n.

9. Show that all the zeroes of $f(z) = z^4 + 6z + 3$ lie inside the circle $|z| = 2$.

10. Evaluate
$$\int_C \frac{1}{(z+i)(z-1)} dz$$
where C is pictured on the next page: