Probability Prelim Exam

January 2020

Instructions (Read before you begin)

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let $X_1, X_2, \ldots,$ and X be real-valued random variables defined on a common probability space. Note they do NOT have to be independent or identically distributed. Prove that
$$\sum_{n=1}^{\infty} \mathrm{E}[|X_n - X|] < \infty \implies X_n \to X \text{ almost surely.}$$

2. Let $A_1, A_2, \ldots,$ be a sequence of events (not necessarily independent) such that
$$\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \infty \quad \text{and} \quad \mathrm{P}(A_n \cap A_m) \leq \mathrm{P}(A_n)\mathrm{P}(A_m) \text{ for } m \neq n.$$
Prove that $\mathrm{P}(A_n \text{ i.o.}) = 1$ or provide a counterexample. (**Hint:** Consider the mean and variance of $\sum_{i=1}^{n} \mathbf{1}_{A_i}$).

3. Let X be a non-negative, integer-valued random variable.
   a) Prove that $\mathrm{E}[X] = \sum_{n=1}^{\infty} \mathrm{P}(X \geq n)$.
   b) A dresser has k distinct pairs of socks (so $2k$ socks total) and the socks are unmatched. We select, at random and without replacement, one sock at a time until a pair has been drawn. Compute the expectation of the total number of draws needed.

4. Let X and Y be any two random variables. Suppose $\mathrm{E}[X^2] < \infty$. The conditional variance of X given Y is defined to be

$$\mathrm{Var}(X \,|\, Y) = \mathrm{E}\big[(X - \mathrm{E}[X \,|\, Y])^2 \,|\, Y\big].$$

Prove the conditional variance formula:

$$\mathrm{Var}(X) = \mathrm{E}\big[\mathrm{Var}(X \,|\, Y)\big] + \mathrm{Var}\big(\mathrm{E}[X \,|\, Y]\big).$$

5. Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of i.i.d. positive integrable random variables. Define $T_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$, and $N(t) = \max\{n \in \mathbb{N} : T_n \leqslant t\}$, $t \geqslant 0$. Show that with probability one, $\frac{N(t)}{t} \to \frac{1}{\mathrm{E}[X_1]}$ as $t \to \infty$. (**Hint:** Sketch what the process $t \mapsto N(t)$ looks like for a fixed realization of the X_n. What do you know about the asymptotic behavior of the jump times?)

6. Fix numbers $p_x, q_x \in (0,1)$, $x \in \mathbb{N}$, such that $p_x + q_x \leqslant 1$. Let $\{Z_{n,x} : n, x \in \mathbb{N}\}$ be independent random variables that take values in $\{-1, 0, 1\}$, with common distribution

$$\mathrm{P}(Z_{n,x} = 1) = p_x, \quad \mathrm{P}(Z_{n,x} = -1) = q_x, \quad \mathrm{P}(Z_{n,x} = 0) = 1 - p_x - q_x.$$

Let $X_0 = 1$ and for $n \in \mathbb{Z}_+$ define inductively $X_{n+1} = X_n + Z_{n,X_n}$ if $X_n \in \mathbb{N}$ and $X_{n+1} = 0$ if $X_n = 0$. In particular, $X_n \in \mathbb{Z}_+$ almost surely. These X_n satisfy

$$\begin{aligned} &\mathrm{P}(X_{n+1} = x + 1 \,|\, X_n = x) = p_x, \\ &\mathrm{P}(X_{n+1} = x - 1 \,|\, X_n = x) = q_x, \\ &\mathrm{P}(X_{n+1} = x \,|\, X_n = x) = 1 - p_x - q_x, \end{aligned}$$

if $x \in \mathbb{N}$ and $\mathrm{P}(X_{n+1} = x \,|\, X_n = x) = 1$ if $x = 0$. The process X_n is called a **birth and death** process, absorbed at 0. Now define the function $\phi : \mathbb{Z}_+ \to \mathbb{R}$ by $\phi(0) = 0$, $\phi(1) = 1$, and for an integer $x \geqslant 2$,

$$\phi(x) = 1 + \sum_{i=1}^{x-1} \prod_{j=1}^{i} \frac{q_j}{p_j}.$$

Prove that $M_n = \phi(X_n)$ is a martingale in the natural filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

7. Assume the same setting as in the previous problem. You can assume that the claim in that problem is true and solve this problem independently.

   a) For an integer $a \geqslant 0$ let $T_a = \inf\{n \geqslant 0 : X_n = a\}$. Fix $a \geqslant 2$ and prove that $P(T_0 < \infty \text{ or } T_a < \infty) = 1$. (**Hint:** As long as the process is not at 0 it has a positive probability to go up a times in a row. How do you turn this observation into a proof?)

   b) Calculate $P(T_a < T_0 \,|\, X_0 = 1)$. (Carefully explain why the conditions of the theorems you use are satisfied.)

8. Use your limit theorems to prove that the following limit exists and to compute its value:
$$\lim_{n\to\infty} \int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{x^2}{2n}\right)^n \, dx.$$
You need to fully justify the use of whatever limit theorem you choose.

9. Recall that the characteristic function of a random variable X is defined as $\phi_X(t) = \mathrm{E}[e^{itX}]$ for $t \in \mathbb{R}$. Prove that if $X \in L^1$ then $\phi'_X(0) = \mathrm{E}[X]$. Make sure to justify all of your steps.

10. Consider the symmetric random variable X taking values in $\mathbb{Z}\backslash\{-1, 0, 1\}$ with distribution
$$\mathrm{P}(X = k) = \mathrm{P}(X = -k) = \frac{C}{k^2 \log k}, \quad k = 2, 3, \ldots$$

    a) Prove that there is a finite $C < \infty$ such that the above is a probability distribution.

    b) Show that $\mathrm{E}[|X|] = \infty$, hence $\mathrm{E}[X]$ is not defined.

    c) However, prove that $\phi'_X(0) = 0$, which shows that the converse direction of the previous problem is not always true. This requires showing that the limit defining $\phi'_X(0)$ exists and equals zero. (**Hint:** Prove then use the inequality $|\cos x - 1| \leq \min(x^2/2, 1)$.)