Probability Prelim Exam

August 2019

Instructions (Read before you begin)

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Assume that A_i, $i = 1, 2, \dots$, are independent events. Prove that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{1}_{A_i} - \mathrm{P}(A_i)\right)$$

converges to zero in probability as $n \to \infty$, where where $\mathbf{1}_{A_i}$ is the indicator function of the event A_i.

2. Let Z_n be a martingale difference with respect to a filtration $\mathscr{F}_n$, i.e. $\mathrm{E}[Z_{n+1}|\mathscr{F}_n] = 0$ for all $n \geqslant 0$. Assume the following hold:

   1) $n^{-1/2}\max_{m\leqslant n}|Z_m| \to 0$ in probability.
   2) $n^{-1}\sum_{m=1}^{n}|Z_m|^2 \to 1$ in probability.
   3) $\sum_{m=1}^{n}|Z_m|^2 \leqslant 2n$ almost surely, for all n.

   Prove that $M_n = n^{-1/2}\sum_{m=1}^{n} Z_m$ converges in distribution to a standard normal.
   Hint: Prove that $|1+ix| \leqslant e^{x^2}$ for all $x \in \mathbb{R}$ and $e^{ix} = (1+ix)e^{-x^2/2+r(x)}$ with $|r(x)| \leqslant x^3$ for all $x \in \mathbb{R}$. Let $T_n = \prod_{m=1}^{n}(1+itn^{-1/2}Z_m)$ and $U_n = e^{-\frac{t^2}{2n}\sum_{m=1}^{n} Z_m^2 + \sum_{m=1}^{n} r(tn^{-1/2}Z_m)}$. Compute $\mathrm{E}[T_n]$ then prove that $\mathrm{E}[T_nU_n] \to e^{-t^2/2}$ and conclude.

3. Let X_k have characteristic function ϕ_k with $X_1, X_2, \dots$, independent random variables. Show that $\sum_{k=1}^{n} X_k$ converges almost surely $\Leftrightarrow$ there exists a neighborhood U of 0 and

a function h such that

$$\prod_{k=1}^{n} \phi_k(u) \xrightarrow{n\to\infty} h(u) \neq 0,$$

for all $u \in U$. **Hint:** For the $\Leftarrow$ direction consider the characteristic function of the partial sums $\sum_{k=m}^{n} X_k$.

4. Let $Z_1, Z_2, \ldots$ be iid with $\mathrm{P}(Z = 1) = \mathrm{P}(Z = -1) = 1/2$, and let $c_1, c_2, \ldots$ be given constants.

   (a) Express the characteristic function of $\sum_{k=1}^{n} c_k Z_k$ in terms of standard elementary functions.

   (b) Show that $\sum_{k\geq 1} c_k Z_k$ converges almost surely $\Leftrightarrow \sum_{k=1}^{\infty} c_k^2 < \infty$. For this you can use results we proved in class, or other problems on this exam.

5. Let (Z_n) be i.i.d. random variables with $P(Z_1 = 1) = P(Z_1 = 2) = 1/2$. Show that $E[Z_1 \cdots Z_n] = (1.5)^n$ and yet $Z_1 \cdots Z_n/(1.45)^n \to 0$ almost surely. Identify the set of numbers $a > 0$ for which $Z_1 \cdots Z_n/a^n \to 0$ almost surely.

6. (Pólya's Urn) An urn initially contains r red and b blue balls. A ball is chosen uniformly at random (i.e. with probability $1/(r + b)$ each). If it comes up red (resp. blue), then it is returned and another red (resp. blue) ball is added to the urn. The process is repeated indefinitely. Let R_n be the number of red balls in the urn after n draws.

   a) A vector $(X_1, X_2, \ldots, X_n)$ of random variables is said to be exchangeable if

   $$(X_1, \ldots, X_n) \sim (X_{\pi(1)}, \ldots, X_{\pi(n)}) \quad \text{for all permutations } \pi,$$

   where $\sim$ denotes equality in distribution. If X_i is the indicator that a red ball is drawn from the urn at time i, prove that $(X_1, \ldots, X_n)$ is exchangeable.

   b) Find the mean and variance of $S_n = X_1 + \ldots + X_n$, the total number of red balls added to the urn up to time n.

7. Let $\{X_{i,n} : i, n \geqslant 1\}$ be i.i.d. random variables with mass function $\{f(x) : x = 0, 1, 2, \ldots\}$. Fix $z \in \{1, 2, \ldots\}$. Define random variables (Z_n) as follows:

   $$Z_0 = z \quad \text{and} \quad Z_n = \sum_{i=1}^{Z_{n-1}} X_{i,n} \quad \text{for } n \geqslant 1.$$

   I.e. Z_n is a branching process with offspring distribution f and initial population z.

   For $s \in [0, 1]$ let $g(s) = \sum_{x=0}^{\infty} f(x)s^x$. Suppose that $s_0 \in (0, 1)$ solves $g(s) = s$. Show that then $s_0^{Z_n}$ is a martingale in the filtration $\mathcal{F}_n = \sigma(X_{i,m} : i \geqslant 1,\, m \leqslant n)$. Use this to conclude that

   $$P(Z_n = 0 \text{ for some } n \geqslant 0) = s_0^z.$$

8. Let X and Y be independent $N(0, 1)$ random variables, and let $Z = X + Y$.

   (a) Show that $\mathrm{E}[Z|X > 0, Y > 0] = 2\sqrt{2/\pi}$.

(b) Find the distribution and the density of Z given that $X > 0$ and $Y > 0$. (You can use the CDF of a standard normal, $\Phi(x) = P(X \leqslant x)$, in your expression.)

9. Suppose $X_1, X_2, \ldots$ are iid with $\mathrm{E}|X_i| = \infty$. Let $S_n = X_1 + \ldots + X_n$, and let A be the event that $M_n = S_n/n$ converges to a finite limit. Let B be the event that $|X_n| \geq n$ infinitely often.

   (a) State the definition of B in terms of unions and intersections.

   (b) Show that $\mathrm{P}(B) = 1$.

   (c) Verify the identity
   $$M_n - M_{n+1} = \frac{M_n}{n+1} - \frac{X_{n+1}}{n+1}$$
   and use it to show that $A \cap B = \varnothing$.

   (d) Use the above to prove that $\mathrm{P}(A) = 0$.

10. Let $X_1, X_2, \ldots$ be independent random variables with
$$X_n = \begin{cases} 1, & \text{with probability } 1/(2n), \\ 0, & \text{with probability } 1 - 1/n, \\ -1, & \text{with probability } 1/(2n). \end{cases}$$
Let $Y_1 = X_1$ and for $n \geq 2$ define
$$Y_n = \begin{cases} X_n, & \text{if } Y_{n-1} = 0, \\ nY_{n-1}|X_n|, & \text{if } Y_{n-1} \neq 0. \end{cases}$$
Show that Y_n is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$. Show that Y_n does not converge almost surely. Why does the martingale convergence theorem not apply?