DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to pass **both** parts.

A. Answer all of the following questions.

1. Give a definition of the flag manifold (or the flag variety) and describe it as a homogeneous space.

2. Prove that the homogeneous space $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is not compact.

3. Give a sketch of the calculation that the Lie bracket on the Lie algebra of $GL_n(\mathbb{R})$, identified with the space of all $n \times n$ matrices, is $[A, B] = AB - BA$.

4. Let M, N, P be smooth manifolds without boundary and $F : M \times P \to N$ a smooth map which is transverse to a submanifold $Q \subset N$. Prove the Transversality Theorem: for almost every $p \in P$ the map $F_p : M \to N$ defined by $F_p(x) = F(x, p)$ is transverse to Q. If you'd like, you can consider the special case when M, Q are submanifolds of $P = N = \mathbb{R}^n$ and $F(x, p) = x + p$.

5. Let $a : S^2 \to S^2$ be the antipodal map on the 2-sphere: $a(x) = -x$, and let

$$p : S^2 \to \mathbb{R}P^2 = S^2/x \sim -x, \quad p(x) = [x]$$

be the projection to the projective plane. Let ω be a 2-form on $\mathbb{R}P^2$.
(a) Prove that $\int_{S^2} p^*\omega = 0$. Hint: $p^*\omega$ is a-invariant.
(b) Prove that there is a 1-form η on $\mathbb{R}P^2$ such that $\omega = d\eta$. You are allowed to use the fact that if ζ is a 2-form on S^2 and $\int_{S^2} \zeta = 0$ then ζ is exact, but you are **not** allowed to use de Rham's theorem.

6. Let $\omega = xdy + dz$ be a 1-form on $\mathbb{R}^3$. Is the 2-plane field $Ker(\omega)$ integrable?

B. Answer all of the following questions.

7. Let $X = S^1 \vee S^1$ and $x_0 \in X$ be the attaching point.
(a) Carefully state van Kampen's Theorem and use it to compute $\pi_1(X, x_0)$.
(b) Find two covering spaces of X with three sheets; one of which is regular (determining a normal subgroup of $\pi_1(X, x_0)$) and the other of which is irregular.

8. What is the universal cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$?

9. Define the singular homology groups $\mathrm{H}_i(X; \mathbb{Z})$ of a topological space X and compute the singular homology groups of the n-sphere S^n, justifying all your steps.

10. Describe a cell structure on real projective space $\mathbb{R}P^n$ and use it to compute all the **cohomology** groups $H^i(\mathbb{R}P^n, \mathbb{Z})$ and $H^i(\mathbb{R}P^n, \mathbb{Z}/2)$.

11. Describe the cohomology **ring** $\mathrm{H}^*(\Sigma_g, \mathbb{Z})$ (i.e. the groups together with the cup product) of a compact oriented surface Σ_g of genus g.
    (a) Prove that if $g < h$, then every map $f : \Sigma_g \to \Sigma_h$ has degree zero.
    (b) Show that for any $g \geq h$, there is a map $f : \Sigma_g \to \Sigma_h$ of degree **one**.

12. Find all cohomology and homology groups (with $\mathbb{Z}$ coefficients) of a closed connected orientable 3-manifold M with $\pi_1(M, m_0) = \mathbb{Z}^r \times G$ where G is a finite group.