DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

January 2012

Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited.

A. Answer all of the following questions.

1. Show that $\mathrm{SL}_n(\mathbb{R})$ is a manifold, and that it is a Lie group. (Do not use the theorem that any closed subgroup of a Lie group is a Lie group.)

2. Prove that a closed, orientable, genus 2 surface does not admit a 1-dimensional foliation.

3. Suppose that a is a smooth, real valued, compactly supported function on $\mathbb{R}^n$ so that $\theta = a dx_1 \wedge \cdots \wedge dx_n$ is an n-form on $\mathbb{R}^n$. What is the definition of $\int_{\mathbb{R}^n} \theta$? What's the definition of $\int_M \omega$ where ω is a compactly supported k-form on an oriented k-dimensional smooth manifold M? Show that this definition is well-defined.

4. State Stokes' Theorem.

5. State Frobenius' Theorem.

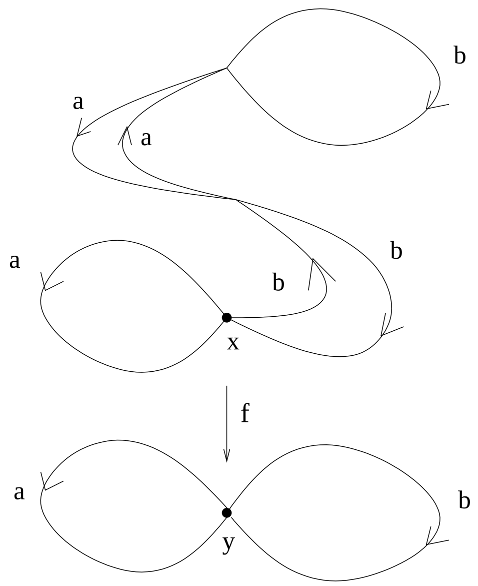
6. Let $f : M \to N$ be a smooth map between smooth manifolds, and let y be a regular value of this map. If M and N are compact, oriented, and have the same dimension, then what is the definition of $deg(f; y)$, the degree of f measured at the value y? We have seen that the degree is independent of the regular value chosen, and an ingredient of that proof is the following lemma:

   Suppose Q is a compact smooth manifold and that $\partial Q = M$. If $F : Q \to N$ is smooth, and z is a regular value of $F|_{\partial Q}$, then $deg(F|\partial Q; z) = 0$.

   Prove the lemma.

B. Answer all of the following questions.

7. Below are pictured two graphs X, Y with basepoints x, y respectively and a covering map $f : X \to Y$ that takes edges to edges preserving labels and orientation.



Identify $\pi_1(Y, y)$ with the free group with basis $\{a, b\}$ as usual. Describe the image of $f_{\#} : \pi(X, x) \to \pi_1(Y, y)$.

8. Let X be the space obtained from the circle S^1 by attaching two 2-cells, one with degree 2 attaching map, and the other with degree 4 attaching map. What is the cardinality of the fundamental group of X? If you use van Kampen's theorem, state it.

9. Using any method you like, compute the homology groups of the space X from Problem 8.

10. Let X be a topological space and $A \subset X$ a subspace. Define relative homology $H_k(X, A)$ and the boundary homomorphism $H_k(X, A) \to H_{k-1}(A)$.

11. State the Excision Theorem for homology.

12. Let X be a topological space and $x \in X$. Assume that for some topological space L the point x has a neighborhood U in X which is homeomorphic to the open cone $cL = L \times [0, \infty)/L \times \{0\}$ by a homeomorphism that takes x to $L \times \{0\}$. Prove that $H_k(X, X - \{x\}) \cong \tilde{H}_{k-1}(L)$.

13. (a) The complex projective space $\mathbb{C}P^n$ can be obtained from $\mathbb{C}P^{n-1}$ by attaching a single $2n$-cell. Describe the attaching map.
    (b) What is $H^k(\mathbb{C}P^n)$ (integral coefficients)? Give a proof by induction on n using (a).
    (c) What is the ring structure on $H^*(\mathbb{C}P^n)$? Give an inductive proof. You may use Poincaré duality.