

UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS  
Ph.D. Preliminary Examination in Differential Equations  
August 19, 2013.

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Instructions: This examination has two parts consisting of six problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth 20 points. A passing score is 72.

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**A. Ordinary differential Equations: Do three problems for full credit**

- A1. Let  $f(x, t)$  be continuous for all  $(x, t) \in \mathbf{R}^n \times \mathbf{R}$  and let  $(x_0, t_0) \in \mathbf{R}^n \times \mathbf{R}$ . Suppose that there is  $M \in \mathbf{R}$  such that

$$|f(x, t) - f(y, t)| \leq M|x - y|, \quad \text{for all } (x, t), (y, t) \in \mathbf{R}^n \times \mathbf{R}.$$

Show that there is and  $\varepsilon > 0$  and a unique function  $y \in \mathcal{C}^1([0, \varepsilon], \mathbf{R}^n)$  that satisfies the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(x, t), \\ x(0) = x_0. \end{cases}$$

- A2. Let  $A$  be an  $n \times n$  real matrix. Consider the linear initial value problem

$$\begin{cases} \frac{dx}{dt} = Ax, \\ x(0) = x_0. \end{cases}$$

Give an algebraic condition on the matrix  $A$  which holds if and only if all solutions of the initial value problem are bounded for all  $t \in \mathbf{R}$ . Prove the equivalence.

- A3. Consider a nonconstant  $T$ -periodic orbit  $\gamma(t)$  of the smooth autonomous differential equation

$$\dot{x} = f(x), \quad x \in \mathbf{R}^n.$$

- (a) Describe the *Poincaré Return Map* associated to  $\gamma$ .  
(b) A  $2\pi$ -periodic solution of the planar system

$$\begin{aligned} \dot{x} &= x - (x + y)\sqrt{x^2 + y^2} \\ \dot{y} &= y + (x - y)\sqrt{x^2 + y^2} \end{aligned}$$

is given by  $\gamma(t) = (\cos t, \sin t)$ . Find a Poincaré Return Map associated to  $\gamma$ .

- (c) Find the derivative of the Poincaré Return Map from part (b) at  $\gamma$ .  
(d) Using (c), determine whether the solution (b) is stable and define the sense of stability you mean.

- A4. Let  $A$  be an  $n \times n$  real matrix all of whose eigenvalues satisfy  $\Re \lambda < 0$ . Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a smooth function such that there is an  $M \in \mathbf{R}$  such that  $|f(x)| \leq M|x|^2$  for all  $x \in \mathbf{R}^n$  so that  $f(0) = 0$ . Consider

$$\begin{cases} \frac{dx}{dt} = Ax + f(x) \cos(\sqrt{1+t}) \\ x(0) = x_0. \end{cases} \quad (1)$$

- (a) Define: the zero solution of (1) is *stable* (also called *Liapunov Stable*).  
 (b) Prove that the zero solution of (1) is stable.

[Don't just quote a theorem. Hint: Suppose  $g(t)$  and  $u(t)$  are nonnegative functions and  $c_0 \geq 0$  is a constant that satisfy  $u(t) \leq c_0 + \int_0^t g(s) u(s) ds$  for all  $t \geq 0$ . Then Gronwall's Inequality implies  $u(t) \leq c_0 \exp\left(\int_0^t g(s) ds\right)$  for all  $t \geq 0$ . ]

- A5. Consider a modified predator-prey model in the first quadrant. Show that it has at least one nonconstant periodic solution.

$$\begin{aligned} \dot{x} &= x(8-x) - \frac{9xy}{x+1}, \\ \dot{y} &= y\left(1 - \frac{y}{x}\right) \end{aligned}$$

- A6. (a) State the Center Manifold Theorem for rest points. Briefly explain its importance in bifurcation theory.  
 (b) Using an approximate center manifold, determine the nature of the bifurcation from  $(x, \dot{x}, \varepsilon) = (0, 0, 0)$  for the quadratic Duffing's Equation

$$\ddot{x} + \dot{x} + \varepsilon x + x^2 = 0.$$

## B. Partial Differential Equations. Do three problems to get full credit

- B1. The small longitudinal free vibrations of an elastic bar are governed by the following equation:

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ E(x) \frac{\partial u}{\partial x} \right].$$

Here  $u$  is the longitudinal displacement,  $\rho$  is the linear density of the material and  $E$  is its Young's modulus. Assume that the bar is constructed by welding together two bars of different (constant) Young's moduli  $E_1, E_2$  and densities  $\rho_1, \rho_2$ , respectively. The displacement  $u$  is continuous across the junction, which is located at  $x = 0$ .

- (a) Give a weak formulation of the global initial value problem, and use this to derive the following jump condition:

$$E_1 u_x(0-, t) = E_2 u_x(0+, t), \quad t > 0.$$

- (b) Let  $c_j^2 = E_j/\rho_j$ . A left incoming wave  $u_I(x, t) = e^{i(t-x/c_1)}$  produces at the junction a reflected wave  $u_R(x, t) = R e^{i(t+x/c_1)}$  and a transmitted wave  $u_T = T e^{i(t-x/c_2)}$ . Determine the reflection and transmission coefficients  $R, T$ , and interpret the result.

- B2. Consider the problem of a thin layer of paint of thickness  $h(x, t)$  and speed  $u(x, y, t)$  flowing down a wall, see Fig. 1. The paint is assumed to be uniform in the  $z$ -direction. The balance between gravity and viscosity (fluid friction) means that the velocity satisfies the equation

$$\frac{\partial^2 u}{\partial y^2} = -c,$$

where  $c$  is a positive constant. This is supplemented by the boundary conditions

$$u(x, 0, t) = 0, \quad \left. \frac{\partial u(x, y, t)}{\partial y} \right|_{y=h} = 0.$$

The density of paint per unit length in the  $x$ -direction is  $\rho_0 h(x, t)$  where  $\rho_0$  is a constant, and the corresponding flux is

$$q(x, t) = \rho_0 \int_0^h u(x, y, t) dy.$$

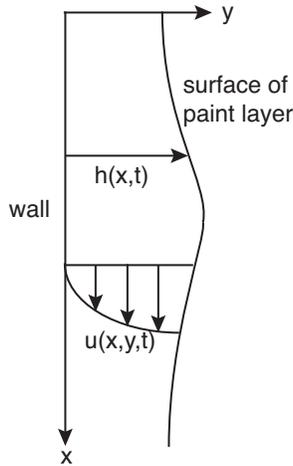


Fig. 1.

- (a) Using conservation of paint, and solving for  $u(x, y, t)$  in terms of  $h(x, t)$  and  $y$ , derive the following PDE for the thickness  $h$ :

$$\frac{\partial h}{\partial t} + ch^2 \frac{\partial h}{\partial x} = 0.$$

- (b) Set  $c = 1$ . Show that the characteristics are straight lines and that the Rankine-Hugoniot condition on a shock  $x = S(t)$  is

$$\frac{dS}{dt} = \frac{[h^3/3]_-^+}{[h]_-^+}.$$

- (c) A stripe of paint is applied at  $t = 0$  so that

$$h(x, 0) = \begin{cases} 0, & x < 0 \text{ or } x > 1 \\ 1, & 0 < x < 1. \end{cases}$$

Show that, for small enough  $t$ ,

$$h = \begin{cases} 0, & x < 0 \\ (x/t)^{1/2}, & 0 < x < t \\ 1, & t < x < S(t) \\ 0, & S(t) < x, \end{cases}$$

where the shock is  $x = S(t) = 1 + t/3$ .

- (d) Explain why this solution changes at  $t = 3/2$ , and show that thereafter

$$\frac{dS}{dt} = \frac{S}{3t}.$$

B3. Consider the following inhomogeneous initial-Neumann problem:

$$\begin{aligned} u_t &= Du_{xx} + \alpha tx, & 0 < x < \pi, \quad 0 < t, \\ u(x, 0) &= 1, & 0 < x < \pi, \\ u_x(0, t) &= u_x(\pi, t) = 0, & 0 < t; \end{aligned}$$

where  $\alpha$  is a constant.

(a) Determine the eigenfunctions of the homogeneous equation

$$u_{xx} = \lambda u, \quad 0 \leq x \leq \pi, \quad u_x(0) = u_x(\pi) = 0.$$

(b) Solve the inhomogeneous initial-Neumann problem by carrying out an eigenfunction expansion of  $u(x, t)$  in terms of the eigenfunctions obtained in part (a). That is, denoting the eigenfunctions by  $v_k$ , integer  $k$ , set

$$u(x, t) = \sum_{k \geq 0} c_k(t) v_k(x),$$

and determine the time-dependent coefficients  $c_k(t)$ .

(c) Give a physical interpretation of the solution in the limit  $\alpha \rightarrow 0$ .

B4. Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad t > 0, \quad x \in \mathbf{R}.$$

along with initial conditions

$$u(x, 0) = g(x), \quad u_t(x, 0) = h(x).$$

(a) Assuming  $c$  is constant, derive d'Alembert's Formula

$$u(x, t) = \frac{1}{2} [g(x + ct) + g(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(\xi) d\xi.$$

(b) Determine the solution for the initial data  $u(x, 0) = 1$  if  $|x| < a$ ,  $u(x, 0) = 0$  if  $|x| > a$ ;  $u_t(x, 0) = 0$

(c) Determine the solution for the initial data  $u(x, 0) = 0$ ;  $u_t(x, 0) = 1$  if  $|x| < a$ ,  $u_t(x, 0) = 0$  if  $|x| > a$

B5. Suppose that  $u(\mathbf{x})$  is a  $C^2$  harmonic function in the domain  $\Omega \subset \mathbf{R}^n$ , so  $\Delta u = 0$  in  $\Omega$ .

(a) Prove the *mean value property*: if  $\mathbf{x} \in \Omega$  and  $r > 0$  is chosen such that  $B_r(\mathbf{x}) \subset \Omega$  (ball of radius  $r$  centered at  $\mathbf{x}$ ) then

$$u(\mathbf{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{s}) d\mathbf{s},$$

where  $\omega_n$  is the measure of  $\partial B_1$ . Hence show that

$$u(\mathbf{x}) \leq \frac{n}{\omega_n r^n} \int_{B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}.$$

(b) Assuming  $\Omega$  is connected, prove that  $u$  can attain its maximum value at an interior point  $x \in \Omega$ , only if  $u$  is constant.