UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 13, 2012.

Instructions: The examination has two parts consisting of five problems in Part A and five in Part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 72.

A. Ordinary differential Equations: Do three problems for full credit

A1. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in t and periodic of period $T > 0$. Consider the linear differential equation in $\mathbb{R}^n$

$$\begin{cases} \dfrac{dx}{dt} = A(t)\,x, \\ x(0) = x_0. \end{cases} \tag{1}$$

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Define: *Floquet Matrix, Floquet Multiplier and Floquet Exponent.* How are these related to $\Phi(t)$? State the necessary and sufficient conditions so that (1) has a nonzero T-periodic solution.

(b) Prove that the zero solution is unstable for the system $\dot{x} = A(t)x$, where

$$A(t) = \begin{pmatrix} 1 & 1 \\ 0 & \dot{h}(t)/h(t) \end{pmatrix},$$

and $h(t) = 2 + \sin t - \cos t$.

A2. Let A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re e\, \lambda_i < 0$ for all $i = 1, \ldots, n$. Consider the initial value problem for $x_0 \in \mathbb{R}^n$,

$$\begin{cases} \dfrac{dx}{dt} = Ax + h(x,t) \qquad \text{with} \qquad h(x,t) = \dfrac{t\|x\|^3}{1+t}\, v, \\ x(0) = x_0 \end{cases} \tag{2}$$

where $v \in \mathbb{R}^n$ is a unit vector.

(a) Show that if $|x_0|$ is small enough, there is a bounded solution to (2) that exists for all time.

[Hint: Using variation-of-parameters and the fact that

$$\lim_{\|x\| \to 0} [\|h(x,t)\|/\|x\|] = 0$$

uniformly in t, for $t > 0$, obtain an integral inequality for $\|x\|$ and then apply Gronwall's Inequality: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \geq 0$ is a constant that satisfy $u(t) \leq c_0 + \int_0^t g(s)\,u(s)\,ds$ for all $t \geq 0$. Then Gronwall's Inequality implies $u(t) \leq c_0 \exp\left(\int_0^t g(s)\,ds\right)$ for all $t \geq 0$.]

(b) Let $z(t) = 0$ for all t be the zero solution to (2). Define what it means for z to be *Lyapunov Stable*. Show that z is Lyapunov stable for this equation.

A3. Consider Griffith's model for a genetic control system, where x and y are proportional to concentration of protein and the messenger RNA from which it is translated, respectively, and $\mu > 0$ is a rate constant

$$\begin{aligned} \dot{x} &= y - \mu x \\ \dot{y} &= \frac{x^2}{1+x^2} - y. \end{aligned}$$

(a) Show that the system has three fixed points when $\mu < \mu_c$ and one when $\mu > \mu_c$ where μ_c is to be determined.

(b) What is the nature of the bifurcation at $\mu = \mu_c$?

A4. Consider a Brusselator system for chemical reactants x, y given by

$$\begin{aligned} \dot{x} &= 1 - 4x + x^2 y, \\ \dot{y} &= 3x - x^2 y \end{aligned}$$

(a) Show that the quadrilateral bounded by the lines $x = 0$, $y = 0$, $y = 8+x$ and $y = 10-x$ is a forward invariant subset. [Hint: need to show that if $\mathbf{n}$ is the outer normal to a given edge of the quadrilateral, then $\mathbf{n} \cdot (\dot{x}, \dot{y}) \leq 0$].

(b) Find the fixed points and determine their stability.

(c) Show that the system has at least one nonconstant periodic solution.

A5. (a) State the Center Manifold Theorem for rest points. Briefly explain its importance in bifurcation theory.

(b) Construct an approximation to the center manifold at the origin and use it to determine the local behavior of solutions.

$$\begin{aligned} \dot{x} &= -xy, \\ \dot{y} &= -y + x^2 - 2y^2. \end{aligned}$$

B. Partial Differential Equations. Do three problems to get full credit.

B1. Consider the second-order equation

$$\nabla^2 u - cu = f \quad \text{in } D \subset \mathbb{R}^n$$

with c constant, $u, f \in C^2(D)$ and

$$u = g \quad \text{on } \partial D,$$

where D is bounded and ∂D is smooth.

(a) Assuming a solution u exists, show that it is unique when $c > 0$. Establish this using two different methods: (i) an integral method and (ii) a Maximum Principle.

(b) Suppose that $n = 2$, $c < 0$, $f = g = 0$, and D is the region $r^2 = x^2 + y^2 < 1$. Show that for a_0 constant,

$$u = a_0 J_0(r\sqrt{-c}),$$

is a solution as long as $\sqrt{-c}$ is a zero of the Bessel function $J_0(x)$, which satisfies

$$\frac{d^2 J_0}{dx^2} + \frac{1}{x}\frac{dJ_0}{dx} + J_0 = 0.$$

(c) Suppose that $n = 3$, $c < 0$, $f = g = 0$, and D is the region $1 < x^2 + y^2 + z^2 < 4$. Show that there are non-trivial solutions if $c = -n^2\pi^2, n = 1, 2, 3 \ldots$.

B2. Suppose that $\rho(x,t)$ is the number density of cars per unit length along a road, x being distance along the road, such that

$$\frac{\partial \rho}{\partial t} + \frac{\partial[\rho(1-\rho)]}{\partial x} = 0.$$

(a) Show that ρ is constant along the characteristics

$$\frac{dx}{dt} = 1 - 2\rho,$$

and derive the following Rankine-Hugoniot condition for the speed of a shock $x = S(t)$:

$$\frac{dS}{dt} = \frac{[\rho(1-\rho)]_-^+}{[\rho]_-^+}.$$

(b) A queue is building up at a traffic light $x = 1$ so that, when the light turns to green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0, & \text{if } x < 0 \text{ and } x > 1; \\ x, & \text{if } 0 < x < 1. \end{cases}$$

Solve the corresponding characteristic equations, and sketch the resulting characteristic curves. Deduce that a collision first occurs at $x = 1/2$ when $t = 1/2$, and that thereafter there is a shock such that

$$\frac{dS}{dt} = \frac{S + t - 1}{2t}.$$

B3. According to Maxwell's Equations, the static magnetic field $\mathbf{H} = (H_1, H_2, H_3)^T$ in an inhomogeneous medium satisfies

$$\nabla \cdot (\mu \mathbf{H}) = 0, \quad \nabla \times \mathbf{H} = 0,$$

where μ is a space-dependent permeability. Suppose the interface $z = 0$ separates a medium in which $\mu = \mu_+$ from one in which $\mu = \mu_-$ with $\mu_\pm$ constants.

(a) Define the weak solution: for every given simply connected, smooth domain $\Omega_3 \subset \mathbb{R}^3$ and arbitrary embedded smooth two dimensional surface with boundary $\Omega_2 \subset \mathbb{R}^3$, assuming enough regularity, integrate by parts the equations

$$\int_{\Omega_3} \psi \nabla \cdot (\mu \mathbf{H})\, dV = 0, \quad \int_{\Omega_2} \psi \nabla \times \mathbf{H} \cdot d\mathbf{S} = 0,$$

where ψ is an arbitrary, suitable test function.

(b) Taking Ω_3 to be a cylindrical domain that cuts the interface vertically, use Green's Theorem to show that

$$[\mu H_3]_-^+ = 0$$

(c) Taking Ω_2 to be a rectangular surface that cuts the interface vertically, use Stokes' Theorem to show that

$$[H_1]_-^+ = 0 = [H_2]_-^+.$$

B4. Let u solve the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad \text{in } \mathbb{R} \times (0, \infty), \\ u = g, \quad u_t &= h, \quad \text{on } \mathbb{R} \times \{0\}. \end{aligned}$$

Suppose g, h have compact support. Define the kinetic energy $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and the potential energy $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$. Prove

(a) The total energy $E = k(t) + p(t)$ is constant in t.

(b) Assume that there are two distinct solutions u, v to the initial value problem. By considering the energy of the difference solution $U = u - v$, prove that $u = v$, that is, the solution is unique.

B5. (a) For a smooth domain $D \subset \mathbb{R}^n$, solve Poisson's equation

$$\nabla^2 u = f \text{ in } D, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial D$$

in terms of an appropriately defined Green's function.

(b) Show that the two-dimensional Green's function for the Laplacian in an unbounded domain is

$$G(\mathrm{x}, \mathrm{x}') = \frac{1}{2\pi} \ln |\mathrm{x} - \mathrm{x}'|.$$

(c) Use the method of images to derive the Green's function for the Laplacian in the half-space $x \in \mathrm{R}$, $y > 0$ with a Neumann boundary condition on $y = 0$, and evaluate the corresponding solution of part (a) when $f = 0$.