DEPARTMENT OF MATHEMATICS
University of Utah

Ph.D. PRELIMINARY EXAMINATION IN DIFFERENTIAL EQUATIONS
January 2007

Instructions: You are to work three problems from part A, and three problems from part B. *Clearly indicate which problems you wish to be graded.*

To receive maximum credit, solutions must be clearly, carefully, and concisely presented and should contain an appropriate level of detail. Each problem is worth 20 points. A passing score is 72.

A. Ordinary Differential Equations

A1. Consider the T–periodic non-autonomous linear differential equation

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^n, \quad A(t) = A(t+T)$$

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = \mathbf{I}$.

(a) Show that there exists at least one nontrivial solution $\mathbf{x} = \chi(t)$ such that

$$\chi(t+T) = \mu\chi(t)$$

where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$\mathbf{x}_i = \mathbf{p}_i(t)\mathrm{e}^{\rho_i t}$$

where the $\mathbf{p}_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(c) Suppose that the autonomous nonlinear equation $\dot{\mathbf{x}} = f(\mathbf{x})$ exhibits a limit cycle. By linearizing about this solution, explain how Floquet theory can be used to determine the linear stability of the limit cycle.

A2. Consider the nonlinear equation

$$\ddot{x} + \varepsilon(\frac{1}{3}\dot{x}^3 - \dot{x}) + x = 0, \quad 0 < \varepsilon \ll 1$$

and choose initial conditions $x(0) = a, \dot{x}(0) = 0$.

(a) Using the method of multiple scales show that this has an asymptotic series solution of the form

$$x(t) \sim 2R(\varepsilon t)\cos(t + \theta(\varepsilon t)) + \mathcal{O}(\varepsilon)$$

with

$$\theta_\tau = 0, \quad R_\tau = \frac{1}{2}R(1 - R^2)$$

where $\tau = \varepsilon t$.

(b) Derive the solution

$$R(\tau) = (1 + a_0 e^{-\varepsilon t})^{-1/2}$$

and determine a_0 from the initial conditions. Hence, establish that there exits a stable periodic orbit.

A3. Consider a second order, linear autonomous system $\dot{x} = Ax, x \in \mathbf{R}^2$.

(a) Suppose that A has a pair of complex conjugate eigenvalues. By performing an appropriate change of coordinates, convert the equation into Jordan normal form and thus obtain the general solution.

(b) Under what conditions is the fixed point at the origin hyperbolic? Explain the significance of hyperbolic fixed points in the theory of nonlinear ODEs.

A4. (a) Use the Poincare–Bendixson theorem to establish the existence of a limit cycle for the system

$$\dot{x} = y + \frac{x}{4}(1 - 2(x^2 + y^2))$$
$$\dot{y} = -x + \frac{y}{2}(1 - (x^2 + y^2)).$$

(b) Consider the system

$$\dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}.$$

Show that the above pair of equations can be rewritten in polar coordinates as

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 2\sin^2(\theta/2)$$

and sketch the phase portrait. Determine whether or not the fixed point $(1, 0)$ is Liapunov stable.

A5. State the center manifold theorem and explain its importance in bifurcation theory. Consider the system

$$\dot{x} = -xy, \quad \dot{y} = -y - x^2.$$

Construct an approximation to the center manifold at the origin and hence determine the local behavior of solutions.

B. Partial Differential Equations.

B1. For each $n \in \mathbb{N}$, consider the Cauchy problem

$$\begin{aligned} -\Delta u_n &= 0, && \text{in } U, \\ u_n &= \frac{1}{n^2}\sin nx, && \text{on } \{(x,y) : y = 0\}, \\ \frac{\partial u_n}{\partial y} &= \frac{1}{n}\sin nx, && \text{on } \{(x,y) : y = 0\}, \end{aligned}$$

where $U = \{(x,y) : 0 < y < 1\}$. Find a sequence $\{u_n\}$ of solutions to these problems, prove that $\{u_n\}$ does not converge to zero, and explain why this implies that the Cauchy problem above is not "well posed".

B2. Let $U \subset \mathbb{R}^n$ be open and bounded, with smooth boundary. For $T > 0$, denote $U_T = U \times (0,T]$. Let $\Gamma_T = \overline{U}_T - U_T$ (closure taken in $\mathbb{R}^n \times \mathbb{R}$). Prove that functions $u(x,t)$ satisfying $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ and

$$\begin{aligned} \Delta u - u &\geq u_t, && \text{in } U_T, \\ u &\geq 0, && \text{in } \overline{U_T}, \end{aligned}$$

also satisfy the maximum principle

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u.$$

B3. Consider the wave equation in three spatial dimensions

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3 \times (0,\infty).$$

(a) Explain the concept of *domain of dependence* for solutions of the wave equation, and describe what this has to do with finite propagation speed.

(b) Sketch a proof, either using a representation formula or energy methods, that solutions $u \in C^2$ of the wave equation have an explicitly defined domain of dependence.

B4. Denote $U = \{x \in \mathbb{R}^2 : x_1 > 0\}$, and $\Gamma = \{x \in \mathbb{R}^2 : x_1 = 1\}$. Use the method of characteristics to solve the first-order problem

$$\begin{aligned} x_1 u_{x_1} + u u_{x_2} - x_2 &= 0 \quad \text{in } U, \\ u(1, x_2) &= 2x_2 \quad \text{on } \Gamma. \end{aligned}$$

(Hint: to solve the characteristic equations, note that the linear combinations $x_2 + z$ and $x_2 - z$ decouple.) Determine the region around Γ upon which the solution is well-defined.

B5. Let $U = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$, and denote $\Gamma = \{|x| = 1\}$. Consider

$$\begin{aligned} -\Delta u &= f, \quad \text{in } U, \\ u &= 0, \quad \text{on } \{x = 0\}, \\ \frac{\partial u}{\partial \eta} &= 0, \quad \text{on } \Gamma. \end{aligned}$$

where η is the unit outward normal, and $f \in L^2(U)$.

Prove a Poincaré-type inequality for this problem, and use it to show that there exists a unique weak solution u in an appropriate Hilbert space.

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A. Ordinary Differential Equations

A1. Consider the T–periodic non-autonomous linear differential equation

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^n, \quad A(t) = A(t+T)$$

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = \mathbf{I}$.

(a) Show that there exists at least one nontrivial solution $\mathbf{x} = \chi(t)$ such that

$$\chi(t+T) = \mu\chi(t)$$

where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$\mathbf{x}_i = \mathbf{p}_i(t)\mathrm{e}^{\rho_i t}$$

where the $\mathbf{p}_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(c) Explain how Floquet theory can be used to determine the linear stability of a limit cycle.

A2. (a) Consider the van der Pol oscillator

$$\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + x = 0, \quad 0 < \varepsilon \ll 1$$

and choose initial conditions $x(0) = 1, \dot{x} = 0$. Using the method of multiple scales show that this has an asymptotic series solution of the form

$$x(t) \sim R(\varepsilon t)\cos(t + \theta(\varepsilon t)) + \mathcal{O}(\varepsilon)$$

with

$$\theta_\tau = 0, \quad R_\tau = \frac{1}{8}R(4 - R^2)$$

where $\tau = \varepsilon t$, $\theta(0) = 0$ and $R(0) = 1$. Hence, establish that there exits a stable periodic orbit.

(b) Consider the Duffing equation

$$\ddot{x} + x = -\varepsilon x^3, \quad 0 < \varepsilon \ll 1.$$

Using the method of multiple scales show that the leading order solution is of the form

$$x(t) \sim R_0 \cos(t + t_0 + \frac{3}{8}R_0^2 \varepsilon t) + \mathcal{O}(\varepsilon).$$

A3. The response of a certain biological oscillator, (x, y), $x \geq 0, y \geq 0$, to a stimulus of size b satisfies the differential equation

$$\begin{aligned} &\dot{x} = x - ay + b, \quad \dot{y} = x - cy \qquad \text{for } x \geq 0, y \geq 0; \\ &\dot{y} = -cy \qquad \text{for } x = 0, \end{aligned}$$

with $a, b, c > 0$.

(a) Using phase–plane analysis show that there exists a limit cycle, part of which lies on the y–axis, when $c < 1$ and $4a > (1 + c)^2$. [Hint: existence of limit cycle depends on existence of an unstable spiral].

(b) Show that the period of the orbit is independent of b.

A4. (a) Give the definitions of Poincare and Liapunov stability. Show that solutions of the system $\dot{x} = y, \dot{y} = 0$ are Poincare but not Liapunov stable.

(b) Consider the system

$$\dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}$$
$$\dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}.$$

Show that the above pair of equations can be rewritten in polar coordinates as

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 2\sin^2(\theta/2)$$

and sketch the phase portrait. Determine whether or not the fixed point $(1, 0)$ is Liapunov stable.

A5. State the center manifold theorem and briefly explain its importance in bifurcation theory. Consider the system

$$\dot{x} = -2x + y - x^2, \quad \dot{y} = x(y - x).$$

Construct an approximation to the center manifold at the origin and hence determine the local behavior of solutions. [Hint: first perform a change of variables in order to diagonalize the linear part of the system].

B. Partial Differential Equations.

B1. Let U be an open subset of $\mathbb{R}^2$. Prove that if $u \in C^2(U)$ satisfies

$$\triangle u - u = 0 \quad \text{in } U,$$

then

$$u(x) = \frac{1}{I(r)} \fint_{\partial B(x,r)} u \, dS_y$$

for every ball $B(x,r) \subset U$, where $I(r) = \frac{1}{\pi}\int_0^\pi e^{r\cos t}\, dt$. Assume without proof that $I(r)$ is the unique solution to the *modified Bessel equation*

$$r\frac{d^2\varphi}{dr^2} + \frac{d\varphi}{dr} - r\varphi = 0, \qquad r > 0,$$

which satisfies $\varphi(0) = 1$.

B2. Let $U \subset \mathbb{R}^n$ be open and bounded, with smooth boundary. For $T > 0$, denote $U_T = U \times (0, T]$. Let $\Gamma_T = \overline{U}_T - U_T$ (closure taken in $\mathbb{R}^n \times \mathbb{R}$). Recall that for solutions $u(x,t)$ of

$$\begin{aligned} u_t - \triangle u &= f, \quad &&\text{in } U_T, \\ u &= g, \quad &&\text{on } \Gamma_T, \end{aligned}$$

we define the *energy* $e(t) = \int_U u^2(x,t)\, dx$.

(a) Use *energy methods* to show that the problem above has at most one solution $u \in C^{2,1}(U_T)$ (C^2 in the x variables, and C^1 in t).

(b) Assuming $g \equiv 0$, and $u \in C^{2,1}(U_T)$, prove that for $0 < t < T$,

$$e(t) \le \left(\int_0^t \left(\int_U f^2(x,s)\, dx \right)^{1/2} ds \right)^2 .$$

B3. Let $\mathbb{R}_+ = (0, \infty)$. Consider the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, && \text{in } \mathbb{R}_+ \times \{t > 0\}, \\ u(x,0) = g(x), \quad u_t(x,0) &= h(x), && x \in \mathbb{R}_+ \quad \text{(initial conditions)}, \\ u_x(0,t) &= 0, && t > 0 \quad \text{(boundary condition)}. \end{aligned}$$

Assume that $g, h \in C^1(\overline{\mathbb{R}}_+)$, and that $g_x(0) = h_x(0) = 0$.

(a) Find an explicit formula for the solution $u(x,t)$ (d'Alembert's formula is a good place to start).

(b) Suppose that for $0 \le t < \frac{1}{2}$ and $x \ge 0$, we know $u(x,t) = g(x+t)$, and assume $\operatorname{supp} g \subset (1,2)$. Thus, initially a wave is travelling toward the origin $x = 0$. What happens to the wave after it hits the origin, say for $t > 2$? Is it absorbed, reflected, damped, inverted? Calculate $u(x,3)$.

B4. Consider the scalar problem

$$\begin{aligned} u_t + H(Du) &= 0, && \text{in } \mathbb{R}^n \times (0,\infty), \\ u &= x_1, && \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned}$$

Assume the Hamiltonian $H(p) = \frac{1}{4}\sum_{j=1}^n \frac{1}{a_j}(p_j - b_j)^2$, where the coefficients a_j, b_j are fixed real numbers, with $a_j > 0$.

(a) Use the Legendre transform $H^*(q) = \sup_p\{p \cdot q - H(p)\}$ to find the Lagrangian.

(b) Setting the coefficients $a_j = 1$, $b_j = 0$ for all j, use the Hopf-Lax formula $u(x,t) = \min_y \left\{tL(\frac{x-y}{t}) + g(y)\right\}$ to find an explicit weak solution to the problem above.

B5. Let $U = \{x \in \mathbb{R}^2 : \frac{1}{2} < |x| < 1\}$, and denote $\partial U = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \{|x| = \frac{1}{2}\}$, $\Gamma_2 = \{|x| = 1\}$. Consider

$$\begin{aligned} -\Delta u &= f, \quad \text{in } U, \\ \alpha u + \frac{\partial u}{\partial \eta} &= 0, \quad \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \eta} &= 0, \quad \text{on } \Gamma_2. \end{aligned}$$

where $\alpha > 0$, η is the unit outward normal on ∂U, and $f \in L^2(U)$.

Prove that there exists a constant $C > 0$ such that

$$C \int_U u^2 \, dx \le \int_U |Du|^2 \, dx + \int_{\Gamma_1} u^2 \, dS$$

for all $u \in C^1(\overline{U})$. Polar coordinates may be helpful. Explain how this estimate could be used in a proof of existence and uniqueness for solutions of the problem above.