UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 14, 2019.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented.
All problems are worth 20 points. A passing score is 72.

A. Ordinary Differential Equations: Do three problems for full credit

A1. (a) Let $f(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. Show that there is an $\epsilon > 0$ and a unique continuously differentiable function $y(t) \in \mathbb{R}^n$ for $t \in [0, \epsilon]$ that satisfies the initial value problem $\frac{dx}{dt} = f(t,x)$, $x(0) = x_0$.

(b) Construct the first few Picard iterates for the differential equation

$$\dot{x} = 2t - \sqrt{max(0,x)}, \quad x(0) = 0. \tag{1}$$

Show that the Picard iterates do not converge. Why do they not converge?

A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in t and periodic of period $T > 0$. Consider the linear differential equation in $\mathbb{R}^n$

$$\begin{cases} \dfrac{dx}{dt} = A(t)\, x, \\ x(0) = x_0. \end{cases} \tag{2}$$

Let $\Phi(t)$ be the fundamental solution matrix with $\Phi(0) = I$.

(a) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$\mathbf{x}_i = \mathbf{p}_i(t) \mathrm{e}^{\rho_i t}$$

where the $\mathbf{p}_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(b) What is the fundamental solution matrix and what are the Floquet exponents for the system

$$A(t) = \begin{pmatrix} a & 1 \\ 0 & \frac{\dot{p}(t)}{p(t)} \end{pmatrix}, \tag{3}$$

where $p(t)$ is a T-periodic function?

A3. Let $p(t) > \alpha > 0$ be a smooth function defined for all t. Prove that for any solution of

$$x'' + p(t)x' + x = 0, \tag{4}$$

$x(t) \to 0$ as $t \to \infty$.

A4. Under what conditions on the parameters a and b does the boundary value problem

$$u'' + u = \sin(x), \quad u(0) = a, \quad u(\pi) = b, \tag{5}$$

have a solution? Is the solution, if it exists, unique? If not, why not?

A5. (a) For a general dynamical system, give the definitions of an *invariant set*, *attracting set* and ω–*limit sets*.

(b) For the system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3,$$

identify all critical points and their stability, and use the Poincare-Bendixson theorem to show that this system has a periodic orbit in the annular region $1 < x^2 + y^2 < 2$.

B. Partial Differential Equations. Do three problems to get full credit

B1. Formulate and prove the maximum principle for Laplace's equation $u_{xx} + u_{yy} = 0$.

B2. Formulate and prove the maximum principle for the heat/diffusion equation $u_t = u_{xx}$.

B3. Show that the heat/diffusion equation with the wrong sign $u_t = -u_{xx}$ is extremely unstable (and so, the Cauchy problem for this equation is ill posed in the sense of Hadamard).

B4. Explain the fundamental difference of wave equations in 1D, 2D, and 3D: In 3D — unlike 1D and 2D — any signal (with finite support) propagates leaving no trace behind it.

B5. Show that a real symmetric linear operator has a full set of real orthogonal eigenvectors (even if it has multiple eigenvalues). [Assume that the operator is a matrix acting in $\mathbb{R}^n$.]