UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 18, 2015.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth 20 points.

A. Ordinary Differential Equations: Do three problems for full credit

A1. Let $f \in C^1(U, \mathbb{R}^n)$ for $U \subset \mathbb{R}^n$ and $x_0 \in U$. Given the Banach space $X = C([0,T], \mathbb{R}^n)$ with norm $\|x\| = \max_{0 \le t \le T} |x(t)|$, let

$$K(x)(t) = x_0 + \int_0^t f(x(s))ds$$

for $x \in X$. Define $V = \{x \in X \mid \|x - x_0\| \le \epsilon\}$ for fixed $\epsilon > 0$ and suppose $K(x) \in V$ (which holds for sufficiently small T), so that $K : V \to V$ with V a closed subset of X.

(a) Give the definition of a locally Lipschitz function in an open set U of a normed vector space.

(b) Using the fact that f is locally Lipschitz in U with Lipschitz constant L_0, and taking $x, y \in V$ show that

$$|K(x(t)) - K(y(t))| \le L_0 t \|x - y\|.$$

Hence, show that

$$\|K(x) - K(y)\| \le L_0 T \|x - y\| \quad x, y \in V.$$

(c) State the contraction mapping principle on a Banach space.

(d) Choosing $T < 1/L_0$, apply the contraction mapping principle to show that the integral equation has a unique continuous solution $x(t)$ for all $t \in [0,T]$ and sufficiently small T. Hence establish existence and uniqueness of the initial value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

A2. Consider the T–periodic non-autonomous linear differential equation

$$\dot{x} = A(t)x, \quad x \in \mathbf{R}^n, \quad A(t) = A(t+T)$$

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = \mathbf{I}$.

(a) Show that there exists at least one nontrivial solution $\chi(t)$ such that

$$\chi(t+T) = \mu \chi(t)$$

where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$x_i = p_i(t)e^{\rho_i t}$$

where the $p_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(c) Consider the equation $\dot{x} = f(t)A_0 x$, $x \in \mathbf{R}^2$, with $f(t)$ a scalar T–periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet multipliers.

A3. Consider the differential operator acting on $L^2(\mathbb{R})$,

$$L = -\frac{d^2}{dx^2}, \quad 0 \leq x < \infty$$

with self-adjoint boundary conditions $\psi(0)/\psi'(0) = \tan\theta$ for some fixed angle θ.

(a) Show that when $\tan\theta < 0$ there is a single negative eigenvalue with a normalizable eigenfunction $\psi_0(x)$ localized near the origin, but none when $\tan\theta > 0$.

(b) Show that there is a continuum of eigenvalues $\lambda = k^2$ with eigenfunctions $\psi_k(x) = \sin(kx + \eta(k))$, where the phase shift η is found from

$$e^{i\eta(k)} = \frac{1 + ik\tan\theta}{\sqrt{1 + k^2\tan^2\theta}}.$$

(c) Evaluate the integral

$$I(x, x') = \frac{2}{\pi}\int_0^\infty \sin(kx + \eta(k))\sin(kx' + \eta(k))dk,$$

and interpret the result with regards the relationship to the Dirac Delta function and completeness, that is, $\delta(x - x') - I(x, x') = \psi_0(x)\psi_0(x')$. You will need the following standard integral

$$\int_{-\infty}^{\infty} e^{ikx}\frac{1}{1 + k^2t^2}\frac{dk}{2\pi} = \frac{1}{2|t|}e^{-|x/t|}.$$

HINT: you should monitor how the bound state contribution (for $\tan\theta < 0$) switches on and off as θ is varied. Keeping track of the modulus signs $|\ldots|$ in the standard integral is crucial for this.

A4. Consider the nonlinear equation

$$\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + x = 0, \quad 0 < \varepsilon \ll 1$$

and choose initial conditions $x(0) = 1, \dot{x}(0) = 0$.

(a) Using the method of multiple scales show that this has an asymptotic series solution of the form

$$x(t) \sim R(\varepsilon t)\cos(t + \theta(\varepsilon t)) + \mathcal{O}(\varepsilon)$$

with

$$\theta_\tau = 0, \quad R_\tau = \frac{1}{8}R(4 - R^2)$$

where $\tau = \varepsilon t$.

(b) Derive the solution

$$R(\tau) = \frac{2}{(1 + a_0 e^{-\varepsilon t})^{1/2}}$$

and determine a_0 from the initial conditions. Hence, establish that there exits a stable periodic orbit. [HINT: perform the change of variable $y = R^2$.]

A5. (a) Show how to convert the ODE

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y).$$

into polar coordinates.

(b) Consider the dynamical system

$$\begin{aligned} \dot{x} &= -y + x(1 - z^2 - x^2 - y^2) \\ \dot{y} &= x + y(1 - z^2 - x^2 - y^2) \\ \dot{z} &= 0. \end{aligned}$$

Determine the invariant sets and attracting set of the system. Give a general definition of the ω-limit set for a flow $\phi(x, t)$ in $\mathbb{R}^n$, and determine it in the case of a trajectory for which $|z(0)| < 1$.

(c) Use the Poincare-Bendixson (PB) Theorem and the fact that the planar system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has only the one critical point at the origin to show that this system has a periodic orbit in the annular region $A = \{x \in R^2 \,|\, 1 < |x| < \sqrt{2}\}$.

B. Partial Differential Equations. Do three problems to get full credit

B1. Consider the problem of a thin layer of paint of thickness $h(x, t)$ and speed $u(x, y, t)$ flowing down a wall, see Fig. 1. The paint is assumed to be uniform in the z-direction. The balance between gravity and viscosity (fluid friction) means that the velocity satisfies the equation

$$\frac{\partial^2 u}{\partial y^2} = -c,$$

where c is a positive constant. This is supplemented by the boundary conditions

$$u(x, 0, t) = 0, \quad \left.\frac{\partial u(x, y, t)}{\partial y}\right|_{y=h} = 0.$$

The density of paint per unit length in the x-direction is $\rho_0 h(x, t)$ where ρ_0 is a constant, and the corresponding flux is

$$q(x, t) = \rho_0 \int_0^h u(x, y, t) dt.$$

(a) Using conservation of paint, and solving for $u(x, y, t)$ in terms of $h(x, t)$ and y, derive the following PDE for the thickness h:

$$\frac{\partial h}{\partial t} + ch^2 \frac{\partial h}{\partial x} = 0.$$

(b) Set $c = 1$. Show that the characteristics are straight lines and that the Rankine-Hugoniot condition on a shock $x = S(t)$ is

$$\frac{dS}{dt} = \frac{[h^3/3]_-^+}{[h]_-^+}.$$

A stripe of paint is applied at $t = 0$ so that

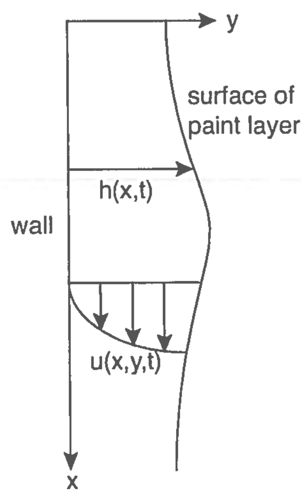
$$h(x,0) = \begin{cases} 0, & x < 0 \text{ or } x > 1 \\ 1, & 0 < x < 1. \end{cases}$$

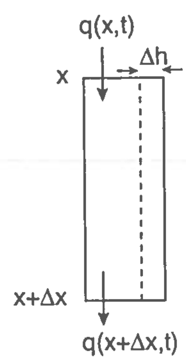
Show that, for small enough t,

$$h = \begin{cases} 0, & x < 0 \\ (x/t)^{1/2}, & 0 < x < t \\ 1, & t < x < S(t) \\ 0, & S(t) < x, \end{cases}$$

where the shock is $x = S(t) = 1 + t/3$. Explain why this solution changes at $t = 3/2$, and show that thereafter

$$\frac{dS}{dt} = \frac{S}{3t}.$$





B2. Suppose that $u(\mathbf{x})$ is a C^2 harmonic function in the domain $\Omega \subset \mathbf{R}^n$, so $\Delta u = 0$ in Ω.

(a) Prove the *mean value property:* if $\mathbf{x} \in \Omega$ and $r > 0$ is chosen such that $B_r(\mathbf{x}) \subset \Omega$ (ball of radius r centered at $\mathbf{x}$) then

$$u(\mathbf{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{s}) ds,$$

where ω_n is the measure of ∂B_1. Hence show that

$$u(\mathbf{x}) \leq \frac{n}{\omega_n r^n} \int_{B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}.$$

(b) Assuming Ω is connected, prove that u can attain its maximum value at an interior point $x \in \Omega$, only if u is constant.

B3. The small longtitudinal free vibrations of an elastic bar are governed by the following equation:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[E(x)\frac{\partial u}{\partial x}\right].$$

Here u is the longtitudinal displacement ρ is the linear density of the material, and E is its Young's modulus. Assume that the bar is constructed by welding together two bars of different (constant) Young's moduli E_1, E_2 and densities ρ_1, ρ_2, respectively. The displacement u is continuous across the junction, which is located at $x = 0$.

(a) Give a weak formulation of the global initial value problem, and use this to derive the following jump condition:

$$E_1 u_x(0-, t) = E_2 u_x(0+, t), \quad t > 0.$$

(b) Let $c_j^2 = E_j/\rho_j$. A left incoming wave $u_I(x,t) = \mathrm{e}^{i(t-x/c_1)}$ produces at the junction a reflected wave $u_R(x,t) = R\,\mathrm{e}^{i(t+x/c_1)}$ and a transmitted wave $u_T = T\,\mathrm{e}^{i(t-x/c_2)}$. Determine the reflection and transmission coefficients R, T, and interpret the result.

B4. (a) Consider a string clamped at the end points a, b with $u(a,t) = u_a$, $u(b,t) = u_b$, where $u(x,t)$ is the string's deviation from the horizontal rest position. The kinetic energy of the string is

$$E_k = \frac{1}{2}\int_a^b \rho u_t^2 ds, \quad ds = \sqrt{1+u_x^2}dx,$$

where $\rho(x,t)$ is the mass density. The potential energy due to stretching of the string is

$$E_p = \int_a^b k\left(\sqrt{1+u_x^2} - 1\right) dx,$$

where $k(x,t)$ is the elastic coefficient.

i. Find the first variation of the action

$$A = \int_0^T (E_k - E_p) dt.$$

ii. Integrating by parts, imposing the boundary conditions, and equating the first variation to zero, derive the PDE

$$\left(\rho u_t \sqrt{1+u_x^2}\right)_t = \left(k(1+u_x^2)^{-1/2} u_x\right)_x.$$

Assuming ρ, k are constants and using the approximation $|u_x| \ll 1$ derive the classical one-dimensional wave equation.

(b) Consider the functional

$$Y[u] = \int_D \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 + fu \right) dx,$$

where D is a bounded domain in $\mathbb{R}^n$ and f is a bounded, continuous function in D. Suppose that there exists a minimizer u^* of $Y[u]$ in the Sobolev space $H_1(D)$. Use this to prove existence and uniqueness of the weak formulation of the PDE (for which $u \in C_1(D)$)

$$-\nabla^2 u + u = -f, \quad x \in D \subset \mathbb{R}^n, \quad \partial_n u = 0, \quad x \in \partial D.$$

[Hint: show that the weak formulation can be expressed as $\langle u, \psi \rangle_{H_1(D)} = 0$ for all $\psi \in H_1(D)$.]

B5. Consider the one-dimensional parabolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^n}{n} \right) = \epsilon \frac{\partial^2 u}{\partial x^2}.$$

(a) Show that a traveling wave solution $u(x,t) = U(z)$ with $z = (x - Vt)/\epsilon$ satisfies

$$\frac{dU}{dz} = \frac{U^n}{n} - VU + \text{constant},$$

and deduce that

$$V = \frac{[U^n/n]_{-\infty}^{\infty}}{[U]_{-\infty}^{\infty}}$$

(b) Discuss how the traveling wave solution relates to shock solutions of the quasilinear equation obtained by setting $\epsilon = 0$. Also show that, when $n = 2$, U can only tend to $U(\pm\infty)$ as $z \to \pm\infty$ if $dU/dz < 0$.