PhD Preliminary Qualifying Examination: Applied Mathematics

Jan. 7, 2010

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A, and the range of the function

$$\phi(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$$

where $x = (x_1, x_2, x_3)$ is a real-valued vector.

(b) Compute $\exp(A)$.

(c) Consider the equation

$$A\,x = \mu\,x$$

Find all solutions for $\mu = 1$ and for $\mu = 3$.

2. Consider linear operator

$$H = -\frac{d^2}{dx^2}\ ,$$

acting on $\psi(x) \in L^2[0, \pi]$ with periodic boundary conditions $\psi(0) = \psi(\pi) = 0$. Find the eigenvalues and eigenfunctions of H. Show that the eigenvectors of H corresponding to distinct eigenvalues are orthogonal. Are the eigenvectors of H complete, and if so, in what sense?

(b) Solve the equation

$$H\ \psi(x) = x^2,$$

representing the solution as an expansion in eigenfunctions of H.

3. Let

$$f(x) = \begin{cases} \frac{100}{n}, & x \text{ rational, } x = \frac{m}{n} \\ -2, & x \text{ irrational.} \end{cases}$$

Does the Riemann integral of $f(x)$ over the interval $[-1,1]$ exist? If so, compute it.

Does the Lebesgue integral of $f(x)$ over the interval $[-1,1]$ exist? If so, compute it.

Why is the Lebesgue integral used in defining the Hilbert space $L^2[0,1]$, and not the Riemann integral?

4. Let

$$f(x) = \text{signum}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0\,. \end{cases}$$

(a) Find its first and second derivatives using the theory of distributions.

(b) Compute

$$\begin{aligned} I_0 &= \int_{-1}^{\infty} \log(x+10)\; f(x)\, dx, \\ I_1 &= \int_{-1}^{\infty} \log(x+10)\; f'(x)\, dx, \\ I_2 &= \int_{-1}^{\infty} \log(x+10)\; f''(x)\, dx. \end{aligned}$$

5. Using Green's functions, solve the problem

$$\frac{d^2u}{dx^2} = \frac{1}{1+x^2}\,, \qquad u(-1) = u(1) = 0$$

for $u(x)$, $x \in [-1,1]$ (obtain an integral representation for the solution, but do not evaluate the integral).

Part B.

1. Find a solution $u(x, y)$ of Laplace's equation on the domain $-\infty < x < \infty$, $0 < y < \infty$ for which $u(x, 0) = x^{1/2}$ for $0 < x < \infty$. What is $u(x, 0)$ for $-\infty < x < 0$?

2. Use Jordan's Lemma (and describe how Jordan's Lemma is used) to evaluate the integral

$$I = \int_0^\infty \frac{\cos ax}{x^2 + 1} dx.$$

3. Use Fourier transforms to solve the integral equation

$$\int_{-\infty}^{\infty} k(x - y)u(y)dy - u(x) = f(x)$$

with $k(x) = H(x)$, the Heaviside function.

4. Use the z-transform to solve the system of equations

$$\frac{du_n}{dt} = \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1}), \qquad -\infty < n < \infty$$

with $u_n(t = 0) = \sin \frac{2\pi n}{k}$, with k an integer.

5. Find the leading order term of the asymptotic expansion of the integral

$$I(s) = \int_{-\infty}^{\infty} e^{is(t+\frac{t^3}{3})} dt, \qquad s \text{ is real and } s \to +\infty,$$

as well as a rigorous estimate of the error.