PhD Preliminary Qualifying Examination:

Applied Mathematics

August 18, 2009

INSTRUCTIONS: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. (a) Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Find the maximum value of the quadratic form $\langle Ax, x \rangle$ for $||x|| \leq 1$. (b) Prove that eigenvectors corresponding to distinct eigenvalues of a self-adjoint matrix are orthogonal. (c) Let $H = -\frac{d^2}{dx^2}$ on $L^2[0,1]$ with boundary conditions $\psi(0) = \psi(1) = 0$. Find the eigenvalues and eigenvectors of H. Show that the eigenvectors of H corresponding to distinct eigenvalues are orthogonal.

2. (a) Define Cauchy sequence, and what it means for a linear space to be complete. (b) Let $\mathcal{C}[0,1]$ be the set of real$-$valued functions which are continuous on $[0,1]$. Show that $\mathcal{C}[0,1]$ is complete under under the uniform norm $||f|| = sup_{x\in[0,1]}|f(x)|$. (c) Let $f_n(x) = x^n$ for $x \in [0,1]$. Find $f = \lim_{n\to\infty} f_n$. Is $f(x)$ continuous? Does this violate the completeness you showed in (a)? Explain.

3. State and prove the Riesz Representation Theorem for a Hilbert space.

4. Let $T : \ell^2 \longrightarrow \ell^2$ be defined by $y = Tx$ with $y_j = x_j e^{-j}$ for $j = 1, 2, 3, ...$, where $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$. Prove that T is a compact operator on ℓ^2.

5. (a) Let $f(x) = |x|$. Find its first and second derivatives using operational calculus, that is, the theory of distributions. (b) Using Green's functions, find the solution to

$$\frac{d^2u}{dx^2} = f(x), \quad u(0) = u(1) = 0.$$

Use the reproducing property of the delta distribution to formally verify that your solution satisfies the equation.

Part B.

1. Formulate and prove the *Maximum Modulus Theorem*.

2. Suppose $f(z)$ is analytic in some neighborhood of the point z_0 without the point z_0 itself; let z_0 be an essential singularity of $f(z)$.

   Show that for any complex number C, there is a sequence of points $z_n \quad (n = 0, 1, 2, ...)$ such that

   $$z_n \to z_0 \quad \text{and} \quad f(z_n) \to C.$$

   (In other words, in every neighborhood of an essential singularity, the function $f(z)$ is arbitrary close to every complex number.)

3. Evaluate the integral

   $$I = \int_{-\infty}^{+\infty} e^{iax^2}\, dx$$

   (a is a positive parameter).

4. Evaluate the integral

   $$I = \int_0^{\infty} \frac{x^\alpha\, dx}{(x^2+1)}$$

   (where α is a parameter, such that the integral converges).

5. Obtain the first two terms of the asymptotic expansion of

   $$I(k) = \int_0^5 \frac{e^{-kt}}{\sqrt{t^2+2t}}\, dx, \qquad k \text{ is real and } k \to +\infty.$$