University of Utah, Department of Mathematics
May 2018, Algebra Qualifying Exam

There are ten problems on the exam. You may attempt as many problems as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Show there is no simple group of order 144.

2. How many elements of order 3 can a group of order 21 have? List all the possibilities.

3. Suppose that $R = \mathbb{C}[x]$, and that M is the R-module generated by three elements a, b, and c, modulo the three relations $a - xc$, $xa - xb + xc$, and $xb - (x^2 - 1)c$. Write M as a direct sum of cyclic modules.

4. Determine the possible characteristic and minimal polynomials of an $n \times n$ matrix over $\mathbb{C}$ that has rank 1.

5. Find all the prime ideals of $\mathbb{Z}[x]/(30,\ x^3 + 1)$ and identify which are maximal.

6. Suppose $R = \mathbb{F}_2[x, y, z]$. Compute $\mathrm{Tor}_1^R(R/(x, y),\ R/(y, z))$.

7. Determine the irreducible factorization of $x^5 - 1$ over the field $\mathbb{F}_{19}$.

8. Let M/K be a Galois extension of degree 700. Prove that there exists an intermediate field L with $[L : K] = 28$. Is L/K necessarily Galois?

9. Determine the Galois group of $x^8 - 2$ over $\mathbb{Q}$.

10. Find a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$.