The Novikov conjecture for algebraic K-theory of the group algebra over the ring of Schatten class operators

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K-Theory

Grothendick, Riemann Roch theorem for algebraic varieties

Atiyah, Hirzebruch, topological K-theory

Whitehead, K_1

Milnor, K_2

Bass, lower algebraic K-theory

Quillen, higher algebraic K-theory

R, a unital ring.

Let $M_\infty(R) = \cup_{n=1}^{\infty} M_n(R)$.

An element $p \in M_\infty(R)$ is called an idempotent if $p^2 = p$.

Example: Let X be a compact space, let $R = C(X)$, the ring of all continuous functions over X.

An idempotent in $M_\infty(C(X))$ corresponds to a vector bundle over X.

Two idempotents in p and q are equivalent if there exists an invertible w in $M_n(R)$ for some large n such that $w^{-1}pw = q$.

Let $Idemp(M_\infty(R))$ be the set of equivalence classes of all idempotents in $M_\infty(R)$.

$Idemp(M_\infty(R))$ is an abelian semi-group with the addition structure:

$$[p] + [q] = [p \oplus q].$$

Definition: $K_0(R)$ is the Grothendick group of the abelian semi-group $Idemp(M_\infty(R))$.

Let $GL_n(R)$ be the group of all invertible matrices in $M_n(R)$, let

$$GL_\infty(R) = \cup_{n=1}^{\infty} GL_n(R).$$

Let $E_n(R)$ be the subgroup of $GL_n(R)$ generated by all invertible matrices in $M_n(R)$, let

$$E_\infty(R) = \cup_{n=1}^{\infty} E_n(R).$$

Basic Fact: $E_\infty(R)$ is the commutator subgroup of $GL_\infty(R)$.

Definition: $K_1(R)$ is the quotient group $GL_\infty(R)/E_\infty(R)$.

Quillen's higher algebraic K-groups: $K_n(R)$

Assume that we have a short exact sequence:

$$0 \to I \to R \to R/I \to 0.$$

If I is H-unital, then there exists a long exact sequence:

$$\cdots \to K_n(I) \to K_n(R) \to K_n(R/I) \to$$

$$K_{n-1}(I) \to K_{n-1}(R) \to K_{n-1}(R/I) \to \cdots .$$

Group ring

Definition: Let Γ be a countable group. Let R be a ring. The group ring $R\Gamma$ is defined to be the ring consisting of all formal finite sum

$$\sum_{\gamma \in \Gamma} r_\gamma \gamma,$$

where $r_\gamma \in R$.

Question: What is $K_n(R\Gamma)$?

Isomorphism Conjecture: The assembly map is an isomorphism:

$$A : H_n^{\Gamma}(E_{VCY}(\Gamma), K(R)^{-\infty}) \longrightarrow K_n(R\Gamma).$$

Here VCY is the family of virtually cyclic subgroups of Γ, $E_{VCY}(\Gamma)$ is the universal Γ-space with isotropy in VCY, $H_n^{\Gamma}(E_{VCY}(\Gamma), K(R)^{-\infty})$ is a generalized Γ-equivariant homology theory associated to the non-connective algebraic K-theory spectrum $K(R)^{-\infty}$.

The isomorphism conjecture is true in the following cases.

Farrell-Jones: fundamental groups of non-positively curved manifolds

Bartels-Lueck: hyperbolic groups

The Novikov conjecture for algebraic K-theory:

The assembly map is rationally injective:

$$A : H_n^{\Gamma}(E\Gamma, K(R)^{-\infty}) \longrightarrow K_n(R\Gamma).$$

Here $E\Gamma$ is the universal Γ-space for free and proper action.

Remark: If the following assembly map is rational injective:

$$A : H_n^{\Gamma}(E_{VCY}(\Gamma), K(R)^{-\infty}) \longrightarrow K_n(R\Gamma),$$

then the algebraic K-theory Novikov conjecture holds for $R\Gamma$.

Theorem (Bokstedt-Hsiang-Madsen): The algebraic K-theory Novikov conjecture holds for $Z\Gamma$ if $H_n(\Gamma)$ if finitely generated for all n, where Z is the ring of integers.

Schatten class operators:

For any $p \geq 1$, an operator T on an infinite dimensional and separable Hilbert space H is said to be Schatten p-class if

$$tr((T^*T)^{p/2}) < \infty,$$

where tr is the standard trace defined by

$$tr(P) = \sum_n < Pe_n, e_n >$$

for any bounded operator P acting on H and an orthonormal basis $\{e_n\}_n$ of H.

For any $p \geq 1$, let S_p be the ring of all Schatten p-class operators on an infinite dimensional and separable Hilbert space.

We define the ring S of all Schatten class operators to be $\cup_{p\geq 1} S_p$.

Connes-Moscovici's higher index theory:

Let M be a compact manifold and D be an elliptic differential operator on M.

The K-theory of the group algebra $S\Gamma$ serves as the receptacle for the higher index of an elliptic operator, i.e.

$$Index(D) \in K_0(S\Gamma)$$

if the dimension of M is even and

$$Index(D) \in K_{-1}(S\Gamma)$$

if the dimension of M is odd.

Main Theorem: The assembly map is rational injective:

$$A : H_n^{\Gamma}(E_{VCY}(\Gamma), K(S)^{-\infty}) \longrightarrow K_n(S\Gamma).$$

Corollary: The Novikov conjecture for algebraic K-theory of $S\Gamma$ holds for all Γ.

“Sketch of Proof”:

Step 1: Reduction to lower algebraic K-theory

(use the Bott element in $K_{-2}(S)$).

Step 2: Use an explicit construction of the Connes-Chern character and its local property to prove that the assembly map is rationally injective.

Open Question 1: Isomorphism conjecture for algebraic K-theory of $S\Gamma$.

Open Question 2: Does the inclusion map induce an isomorphism:

$$i_* : K_n(S\Gamma) \to K_n(K \otimes C_r^*(\Gamma))?$$

here K is the algebra of all compact operators on a separable and infinite dimensional Hilbert space.

Open Question 3: Is i_* rationally injective?

A positive answer of the above question would imply the Novikov higher signature conjecture.