Topological Properties of Diffeomorphism Groups of Non-Compact Manifolds

Tatsuhiko Yagasaki

(Kyoto Institute of Technology, Japan)

Dubrovnik VII – Geometric Topology
Inter-University Centre in Dubrovnik
Dubrovnik, Croatia
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§.1. Main Problem

Local and Global Topological Properties of

Diffeomorphism Groups of **Non-Compact** C^∞ n-Manifolds

Typical Topologies :

(1) **The Compact-Open C^∞ Topology**

(2) **The Whitney Topology**

(Joint Works with T. Banakh, K. Mine and K. Sakai)

Compact-Open C^∞ Topology $\longleftrightarrow$ Tychonoff Products • Weak Products of ℓ_2

Whitney C^∞ Topology $\longleftrightarrow$ Box Products • Small Box Products of ℓ_2

Notations

M : Connected C^∞ n-Manifold

$\mathcal{D}(M)$ = Group of Diffeomorphisms of M

When M has Volume form ω,

$\mathcal{D}(M;\omega)$ = Subgroup of ω-Preserving Diffeomorphisms of M

$\mathcal{G}(M) \subset \mathcal{D}(M)$: any subgroup

(1) $\mathcal{G}^+(M)$: Orientation - Preserving

$\mathcal{G}^c(M)$: Compact Support

(2) when $\mathcal{G}(M)$ is endowed with a topology

$\mathcal{G}(M)_0$: Connected Component of id_M in $\mathcal{G}(M)$

$\mathcal{G}(M)_1$: Path Component of id_M

(3) $\mathcal{G}^c(M)_1 \supset \mathcal{G}^c(M)_1^* = \left\{ h \in \mathcal{G}^c(M) \;\middle|\; \begin{array}{r} \exists \text{ a path } h \overset{h_t}{\simeq} \mathrm{id}_M \text{ in } \mathcal{G}^c(M) \\ \text{with Common Compact Support} \end{array} \right\}$

§.2. Properties of Compact-Open C^∞ Topology

When M is Compact $\qquad \mathcal{D}(M)$: Fréchet manifold $\quad$ (Top l_2-manifold)

$$\begin{array}{ccc} \mathcal{D}^+(M) & \supset & \mathcal{D}(M)_0 \\ \textbf{SDR} \cup & & \textbf{SDR} \cup \\ \mathcal{D}(M;\omega) & \supset & \mathcal{D}(M;\omega)_0 \end{array} \quad \Longleftarrow \quad \begin{array}{c} \text{Parametrized version of} \\ \textbf{Moser's Thm} \end{array}$$

When M is Non-Compact

$$c_0^\omega : \mathcal{D}(M;\omega)_0 \longrightarrow \mathcal{S}(M;\omega)$$

End - Charge Homo.

Theorem 2.1.

$$\begin{array}{ccccccccc} \mathcal{D}^+(M) & \supset & \mathcal{D}(M)_0 & & \supset & & \mathcal{D}^c(M)_0 & \supset & \mathcal{D}^c(M)_1^* \\ \cup & & \textbf{SDR} \cup & \textbf{SDR} & & & \cup & & \cup \\ \mathcal{D}(M;\omega) & \supset & \mathcal{D}(M;\omega)_0 & \supset & \ker c_0^\omega & \supset & \mathcal{D}^c(M;\omega)_0 & \supset & \mathcal{D}^c(M;\omega)_1^* \end{array}$$

Parametrized version of Moser's Thm for Non-Compact Manifolds

Continuous Section of End - Charge Homo.

$\Longleftarrow$ Moser's Thm for Compact Manifolds

+ Realization of data of transfer of Volume toward Ends by Diffe's

Remaining Problems

$$\begin{array}{ccccccc}
\mathcal{D}(M)_0 & & & \supset & & & \mathcal{D}^c(M)_1^* \\
\textbf{SDR}\ \cup & & & & & & \cup \\
\mathcal{D}(M;\omega)_0 & \overset{\textbf{SDR}}{\supset} & \ker c_0^\omega & & \supset & & \mathcal{D}^c(M;\omega)_1^*
\end{array}$$

[1] Homotopy / Topological Type of $\mathcal{D}(M)_0$ and $\mathcal{D}(M;\omega)_0$

[2] Relations between $\mathcal{D}(M)_0 \supset \mathcal{D}^c(M)_1^*$

$\ker c_0^\omega \supset \mathcal{D}^c(M;\omega)_1^*$

In $n = 2$ we can answer these questions.

2-dim case

M : Non-Compact Connected C^∞ 2-Manifold without Boundary

Exceptional Case — Plane, Open Möbius Band and Open Annulus

Generic Case — All Other cases

[1] Homotopy / Topological Type of $\mathcal{D}(M)_0$ and $\mathcal{D}(M;\omega)_0$

Theorem 2.2.

(1) $\mathcal{D}(M)_0$: Topological ℓ_2-Manifold

(2) (i) Generic Case : $\mathcal{D}(M)_0 \simeq *$ $\mathcal{D}(M)_0 \approx \ell_2$

(ii) Exceptional Case : $\mathcal{D}(M)_0 \simeq \mathbb{S}^1$ $\mathcal{D}(M)_0 \approx \ell_2 \times \mathbb{S}^1$

Theorem 2.3.

(1) $\mathcal{D}(M;\omega)_0$: Topological ℓ_2-Manifold

(2) (i) Generic Case : $\mathcal{D}(M;\omega)_0 \approx \ell_2$

(ii) Exceptional Case : $\mathcal{D}(M;\omega)_0 \approx \ell_2 \times \mathbb{S}^1$

[2] Subgroups $\mathcal{D}^c(M)_1^*$ and $\mathcal{D}^c(M;\omega)_1^*$

Remark on $\mathcal{D}^c(M)_1^*$

(1) By Definition,

each $h \in \mathcal{D}^c(M)_1^*$ is Isotopic to id_M with Compact Support.

(2) However,

a path in $\mathcal{D}^c(M)_1^*$ need **not** have Common Compact Support.

∘ $h_i \longrightarrow \mathrm{id}_M$ in $\mathcal{D}^c(M)_1^*$ if $\mathrm{Supp}\, h_i \longrightarrow \infty$.

Example 1. $M = \mathbb{R}^2$

(1) $\mathcal{D}(\mathbb{R}^2)_0 \simeq \mathbb{S}^1$ This homotopy equivalence is induced from

Loop of θ rotations $\varphi(\theta)$ $(\theta \in [0, 2\pi])$

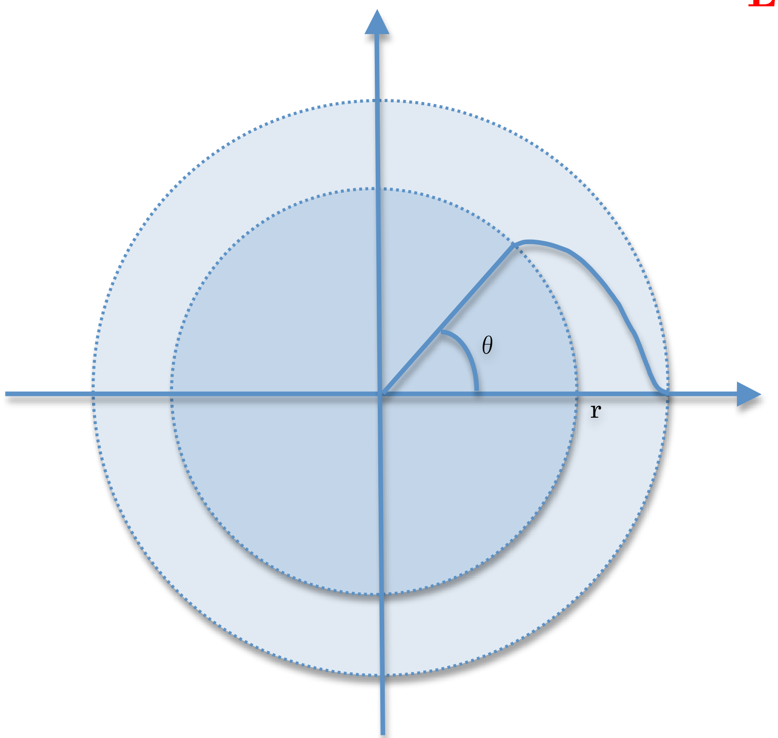
(2) $\mathcal{D}^c(\mathbb{R}^2)_1^* = \mathcal{D}^c(\mathbb{R}^2) \subset \mathcal{D}(\mathbb{R}^2)_0$

False: $\mathcal{D}^c(\mathbb{R}^2) \simeq *$ by Alexander Trick

$\exists$ **Deformation** $\varphi_t(\theta)$ $(t \in [0,1])$ of $\varphi_0(\theta) \equiv \varphi(\theta)$

$$\varphi_t(\theta) \in \mathcal{D}^c(\mathbb{R}^2) \text{ for } 0 < t \leq 1.$$

We can take $\varphi_t(\theta)$ $(\theta \in [0, 2\pi])$ as **Loop of Truncated θ rotations**



Level of Truncation $r = r_t(\theta)$

$r = r_t(\theta)$ need to satisfy :

(i) $r \to \infty$ as $\theta \to 2\pi$ (for each $t > 0$)

(ii) $r \to \infty$ uniformly as $t \to 0$

$\mathcal{D}^c(\mathbb{R}^2) \subset \mathcal{D}(\mathbb{R}^2)_0$: HE

$\not\simeq$

$*$

Example 2. M = Open Annulus

h : Dehn Twist on M along Center circle of M

(1) $h \in \mathcal{D}(M)_1 \setminus \mathcal{D}_c(M)_1^*$

(2) $\exists$ a path $h \overset{h_t}{\simeq} \mathrm{id}_M$ in $\mathcal{D}(M)_0$ s.t. $h_t \in \mathcal{D}_c(M)_1^*$ $(0 < t \leq 1)$

(Introduce Reverse Dehn Twist from ∞)

Definition 2.1. $A \subset X$: **Homotopy Dense (HD)**

$$\Longleftrightarrow \ \exists\, \varphi_t : X \to X : \text{Homotopy s.t. } \ \varphi_0 = id_X, \ \ \varphi_t(X) \subset A \ (0 < t \leq 1)$$

Theorem 2.4. $\mathcal{D}(M)_0 \supset \mathcal{D}^c(M)_1^* :$ Homotopy Dense

$$\circ \ \ \mathcal{D}(M)_0 \supset \mathcal{D}^c(M)_0 = \mathcal{D}^c(M)_1 \supset \mathcal{D}^c(M)_1^* : \text{Homotopy Equi.}$$

$\left(\prod^\omega \ell_2, \sum^\omega \ell_2\right)$ **:** $\quad \sum^\omega \ell_2 = \left\{(x_i)_i \in \prod^\omega \ell_2 \mid x_i = 0 \text{ except for finitely many } i\right\}$

$$\prod^\omega \ell_2 \cong \ell_2$$

Corollary 2.1. $\mathcal{G} = \mathcal{D}^c(M)_0$ or $\mathcal{D}^c(M)_1^*$

(1) $\left(\mathcal{D}(M)_0, \mathcal{G}\right)$: Topological $\left(\prod^\omega \ell_2, \sum^\omega \ell_2\right)$-manifold

(2) (i) Generic Case : $\qquad \left(\mathcal{D}(M)_0, \mathcal{G}\right) \approx \left(\prod^\omega \ell_2, \sum^\omega \ell_2\right)$

(ii) Exceptional Case : $\qquad \left(\mathcal{D}(M)_0, \mathcal{G}\right) \approx \left(\prod^\omega \ell_2, \sum^\omega \ell_2\right) \times \mathbb{S}^1$

Theorem 2.5. $\ker c_0^\omega \supset \mathcal{D}^c(M;\omega)_1^* :$ Homotopy Dense

$$\circ \ \ \ker c_0^\omega \supset \mathcal{D}^c(M;\omega)_0 = \mathcal{D}^c(M;\omega)_1 \supset \mathcal{D}^c(M;\omega)_1^* : \text{Homo. Equi.}$$

§3. Properties of Whitney C^∞-Topology

(Joint Work with T. Banakh, K. Mine and K. Sakai)

M : Connected C^∞ n-manifold without Boundary

$\mathcal{D}(M)^w = \mathcal{D}(M)$ with Whitney C^∞-topology

∘ $h \in \mathcal{D}(M)^w$ has Basic Nbds of the following form :

$$\bigcap_{\lambda\in\Lambda} \mathcal{U}(h,\ (U_\lambda, x_\lambda),\ (V_\lambda, y_\lambda),\ K_\lambda,\ r_\lambda,\ \varepsilon_\lambda)$$

where $\{U_\lambda\}_{\lambda\in\Lambda}$: Locally Finite in M

When M is Compact Whitney C^∞-Top = Compact-Open C^∞-Top

When M is Non-Compact

Whitney C^∞-Top : Too Strong (Compact-Open C^∞-Top : Too Weak)

(1) $\mathcal{D}(M)^w_0 = \mathcal{D}^c(M)^*_1 \subset \mathcal{D}^c(M)$ (as Sets)

(2) Any compact subset in $\mathcal{D}^c(M)^w$ has Common Compact Support.
(any path)

Local Top Type of $\mathcal{D}(M)^w$ and $\mathcal{D}^c(M)^w$

$$\mathbb{R}^\infty = \varinjlim_n \mathbb{R}^n$$

Theorem 3.1.

(1) $\mathcal{D}^c(M)^w$: Paracompact $(\ell_2 \times \mathbb{R}^\infty)$-manifold

(2) (i) $\mathcal{D}(M)^w_0 \subset \mathcal{D}^c(M)^w$: Open Normal Subgroup

(ii) $\mathcal{M}_c(M) := \mathcal{D}^c(M)^w / \mathcal{D}(M)^w_0$ (Discrete Countable Group)

$\mathcal{D}^c(M)^w \approx \mathcal{D}(M)^w_0 \times \mathcal{M}_c(M)$ (as Top Spaces)

(3) $(M_i)_{i\in\mathbb{N}}$: Sequence of Compact C^∞ n-Submanifolds of M

s.t. $M_i \subset \mathrm{Int}_M M_{i+1}$, $M = \cup_i M_i$

$G(M_i) := \{h \in \mathcal{D}^c(M)^w \mid \operatorname{supp} h \subset M_i\}$

$\Longrightarrow$ $\mathcal{D}^c(M)^w = \text{g-}\varinjlim_i G(M_i)$ (Direct Limit in Category of Top Groups)

∘ $\mathcal{D}^c(M)$ with **Direct Limit Top** : **Not Top Group**

Theorem 3.2. $(\mathcal{D}(M)^w, \mathcal{D}^c(M)^w) \approx_\ell (\Box^\omega, \boxdot^\omega)\, l_2$ at id_M

Global Top Type of $\mathcal{D}^c(M)^w$

Theorem 3.3.

(1) $n = 1:\ \ (\mathcal{D}(\mathbb{R})^w, \mathcal{D}^c(\mathbb{R})^w) \approx (\square^\omega, \boxdot^\omega)\ l_2$

(2) $n = 2:\ \ \mathcal{D}(M)_0^w \approx l_2 \times \mathbb{R}^\infty$

(3) $n = 3:\ \ M$: Orientable, Irreducible $\implies \mathcal{D}(M)_0 \approx l_2 \times \mathbb{R}^\infty$

(4) X : Compact Connected C^∞ n-manifold with Boundary

$$M = \mathrm{Int}\, X \implies \mathcal{D}(M)_0^w \approx \mathcal{D}(X, \partial X)_0^w \times \mathbb{R}^\infty$$

In $n = 2$ $\quad \mathcal{M}_c(M) = \mathcal{D}^c(M)^w / \mathcal{D}(M)_0^w$

S : Connected 2-manifold possibly with Boundary

(1) S : **Exceptional** $\iff S \approx N - K$, where

$N =$ Annulus, Disk or Möbius band

$K =$ Non-empty Compact Subset of One Boundary Circle of X

(2) S : **Semi-Finite Type**

$$\overset{\text{def}}{\Longleftrightarrow} \ S \approx N - (F \cup K) \ \text{ s.t. } \ \begin{array}{l} N : \text{Compact connected 2-manifold} \\ F \subset N \setminus \partial N : \text{a finite subset} \\ K \subset \partial N : \text{a compact subset} \end{array}$$

$$\overset{\text{equiv.}}{\Longleftrightarrow} \ \pi_1(S) : \text{finitely presented} \ \overset{\text{equiv.}}{\Longleftrightarrow} \ H_1(S;\mathbb{Z}) : \text{finitely generated}$$

Proposition 3.1. The Following Conditions are Equivalent:

(1) $\mathcal{M}_c(S) = \{1\}$ (2) $\mathcal{M}_c(S)$: Torsion Group

(3) $\operatorname{asdim} \mathcal{M}_c(S) = 0$ (4) S : Exceptional

Proposition 3.2. The Following Conditions are Equivalent:

(1) $\mathcal{M}_c(S)$: finitely generated (or finitely presented)

(2) $r_{\mathbb{Z}} \mathcal{M}_c(S) < \infty$ (3) $\operatorname{asdim} \mathcal{M}_c(S) < \infty$

(4) S : Semi-Finite Type

- S : Not Semi-Finite Type $\Longrightarrow$ $\mathbb{Z}^\infty \subset \mathcal{M}_c(S)$

(Free Abelian Group of Infinite Rank)

Idea of Proofs.

M : Non-compact C^∞ n-manifold without boundary

Notations

(a) $G_K(N) = \{g \in \mathcal{D}(M)^w \mid g = \mathrm{id} \text{ on } K \text{ and } M \setminus N\}$ $(K, N \subset M)$

(b) Space of Embeddings : $(L \subset M)$

$$\mathcal{E}^G(L, M) = \{h|_L \mid h \in \mathcal{D}(M)^w\} \quad \text{(Compact - Open } C^\infty \text{ Top)}$$

$(i_L : L \subset M$: the base point)

$(M_i)_{i \in \mathbb{N}}$: Sequence of Compact n-Submanifolds of M

s.t. $M_i \subset \mathrm{Int}_M M_{i+1}$, $M = \bigcup_i M_i$

(1) $p : \boxdot_i G(M_i) \longrightarrow \mathcal{D}^c(M)^w$: the multiplication map

$\Longrightarrow$ p has a local section. $\mathcal{D}^c(M)^w = \text{g-}\varinjlim_i G(M_i)$

$L_i := M_i - \mathrm{Int}_M M_{i-1}$ $(M_0 = \emptyset)$

(2) $$\begin{aligned}(\mathcal{D}(M)^w, \mathcal{D}_c(M)^w) &\approx_\ell (\square, \boxdot)_i \, \mathcal{E}^G(L_{2i}, M) \times (\square, \boxdot)_i \, G(L_{2i-1}) \\ &\approx_\ell (\square, \boxdot)^\omega \, l_2 \qquad \text{at } \mathrm{id}_M\end{aligned}$$

(3) In $n = 2$

Apply Theorem A to $\mathcal{D}(M)_0^w$ and $G_i = G(M_i)_0$ $(i \in \mathbb{N})$:

$$\begin{tikzcd}
 & \boxdot_i G_i \arrow[dr, "q"] & \\
\boxdot_i (G_i/G_{i-1}) \arrow[ur, "s = \boxdot_i s_i"] \arrow[rr, "qs", "\approx"'] & & \mathcal{D}(M)_0^w
\end{tikzcd}$$

$$\mathcal{D}(M)_0^w \approx \boxdot_i (G_i/G_{i-1}) \approx \boxdot_i \ell_2 \approx \ell_2 \times \mathbb{R}^\infty$$

Remark.

In (3) we can also apply

Topological Characterization of $\ell_2 \times \mathbb{R}^\infty$

by T. Banakh - D. Repovš (arXiv:0911.0609) (2009 -)

Appendix. Box products and Small box products

A-1. Definitions and Basic Properties

$(X_n)_n$: Box product : $\Box_n X_n$ = Product $\prod_n X_n$

Box Top : $\prod_n U_n$ ($U_n \subset X_n$: Open subset)

$(X_n, *_n)_n$: Small box product : $\boxdot_n(X_n, *_n) \subset \Box_n X_n$

$$(x_0,\ x_1,\ \ldots,\ x_k,\ *_{k+1},\ *_{k+2},\ \ldots)$$

$(\Box, \boxdot)_n X_n = (\Box_n X_n, \boxdot_n X_n)$ $\qquad$ $(\Box, \boxdot)^\omega X = (\Box, \boxdot)_{n \in \omega} X$

(1) X_n : metrizable $\Longrightarrow$ $\boxdot_n X_n$: paracompact

(2) $\boxdot^\omega \ell_2 \approx \ell_2 \times \mathbb{R}^\infty$ Top. Classification of LF spaces by P. Mankiewicz

(3) $\Box^\omega \ell_2$: not locally connected, not normal

A-2. Small box products of Top Groups

(1) $(G_n)_{n\in\omega}$: Top Groups $\quad (e_n$: the identity element of $G_n)$

$\square_n G_n$: Top Group

$\boxdot_n (G_n, e_n) \subset \square_n G_n$: Top Subgroup

(2) G : Top group $\quad (e$: the identity element of $G)$

$(G_n)_{n\in\omega}$: Increasing Sequence of Subgroups of G s.t. $G = \bigcup_n G_n$

Multiplication maps

$$p : \boxdot_n G_n \longrightarrow G : \quad p(x_0, x_1, \ldots, x_k, e, e, \ldots) = x_0 x_1 \cdots x_k$$

$$q : \boxdot_n G_n \longrightarrow G : \quad q(x_0, x_1, \ldots, x_k, e, e, \ldots) = x_k \cdots x_1 x_0$$

(i) p, q : Continuous, Surjective

(ii) p : Open at $(e)_n \quad \Longrightarrow \quad G = \text{g-}\varinjlim_n G_n \quad$ (in Category of Top Groups)

$\Updownarrow$

q : Open at $(e)_n$

Theorem A. (BMSY, arXiv:0802.0337v1 (2007 - 2008))

G : Top group

(G_i) : Increasing Sequence of Closed Subgroups of G s.t. $G = \bigcup_i G_i$

$q : \boxdot_i G_i \to G : q(x_1, \ldots, x_m, e, e, \cdots) = x_m \cdots x_1$

$(*_1)$ $q : \boxdot_i G_i \to G$: Open

$(*_2)$
$$\begin{tikzcd} \pi_i : G_i \arrow[r] & G_i/G_{i-1} \arrow[l, bend left, "\exists\ s_i"] \end{tikzcd} \qquad \pi_i s_i = \mathrm{id}$$

$\Longrightarrow$
$$\begin{tikzcd} & \boxdot_i G_i \arrow[dr, "q"] & \\ \boxdot_i (G_i/G_{i-1}) \arrow[ur, "s = \boxdot_i s_i"] \arrow[rr, "qs", "\approx"'] & & G \end{tikzcd}$$

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End of Talk

Thank you very much !