Hairy graphs and the cohomology of Out(F_n)

James Conant, Martin Kassabov, Karen Vogtmann

Cyclic operads, Lie algebras and homology

Give me your favorite cyclic operad and I will make a Lie algebra, à la Kontsevich.

For at least some cyclic operads $\mathcal{O}$, the homology of this Lie algebra is interesting for topologists/geometric group theorists

For $\mathcal{O}$ = Assoc, it computes the cohomology of punctured mapping class groups [Kontsevich].

For $\mathcal{O}$ = Lie, it computes the cohomology of Out(F_n) [Kontsevich].

This talk is about a new way to construct cycles for the homology of these Lie algebras, using hairy graphs.

As an application, we find new cohomology classes for Out(F_n)

Outline of talk

(1) Cyclic operads

(2) The associated Lie algebra ℓ_∞

(3) Symplectic invariants and Kontsevich's theorem

(4) The hairy graph complex and embedding $H_*(\ell_\infty) \hookrightarrow H_*(\mathcal{H})$

(5) Compute $H_1(\ell_\infty)$ for $\mathcal{O}$ = Assoc

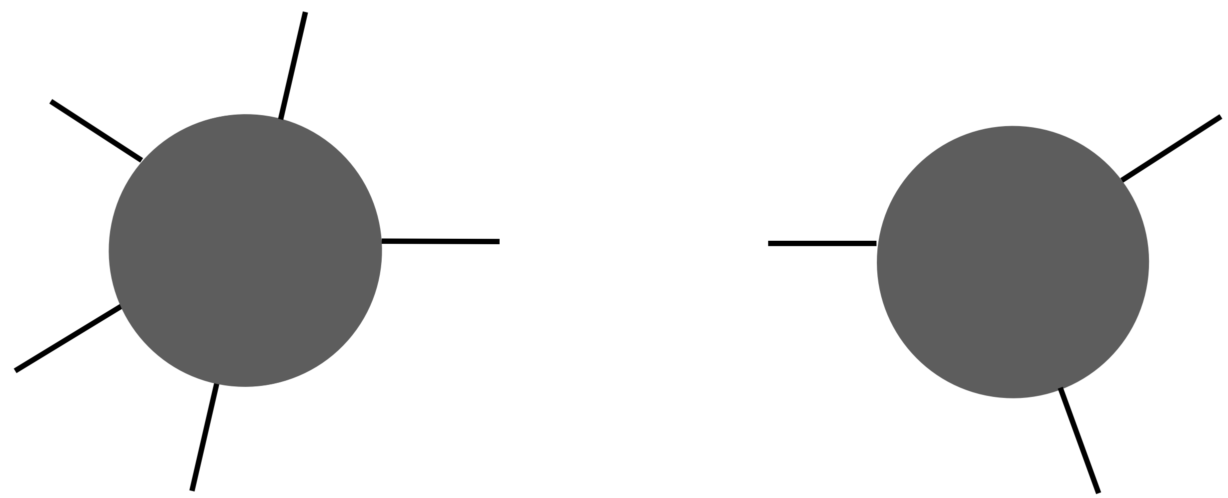
(6) Compute $H_1(\ell_\infty)$ for $\mathcal{O}$ = Lie

(7) Note the connection with modular forms.

(8) New cohomology classes for $Out(F_n)$

Cyclic operads

Generated as a vector space by black boxes with numbered i/o slots:



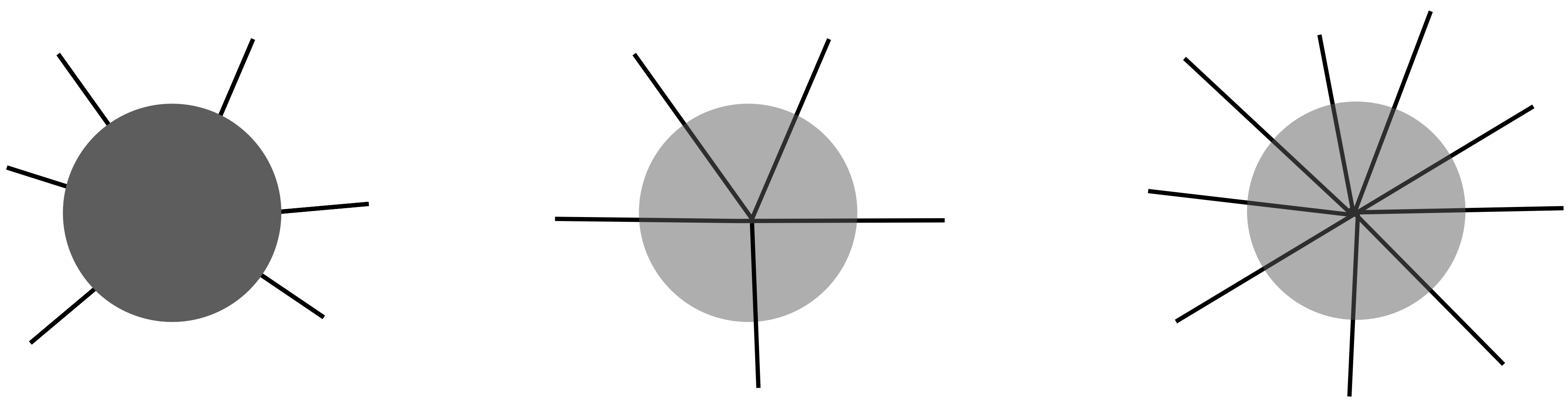
There is an operation merging boxes using an i/o slot from each

This operation obeys a very long list of natural rules including associativity, the existence of an identity and rules governing the interaction with the symmetric group action on i/o slot numbers.

Cyclic operads

EXAMPLES:

(1) Put a planar “star” in the black box

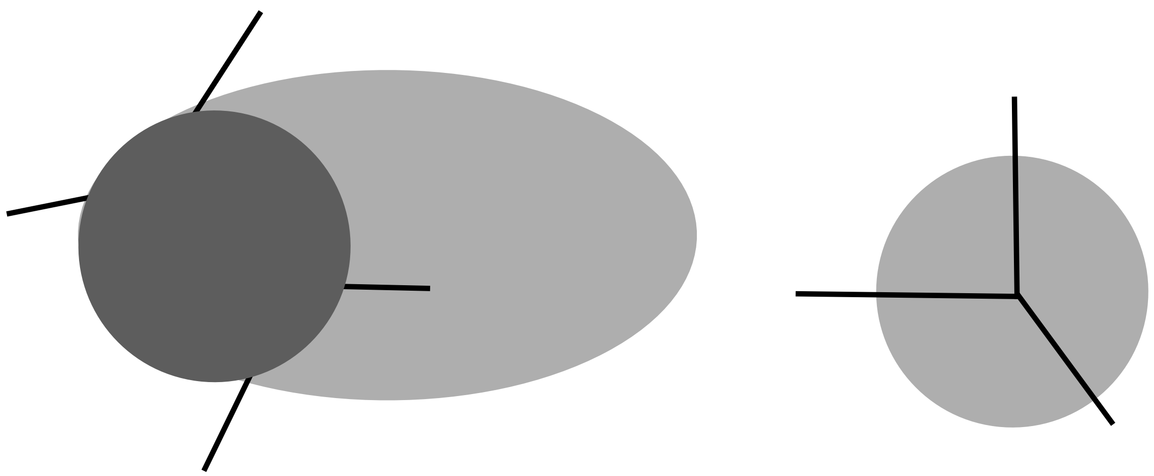


Merge by joining two leaves, then collapsing the resulting edge.

Cyclic operads

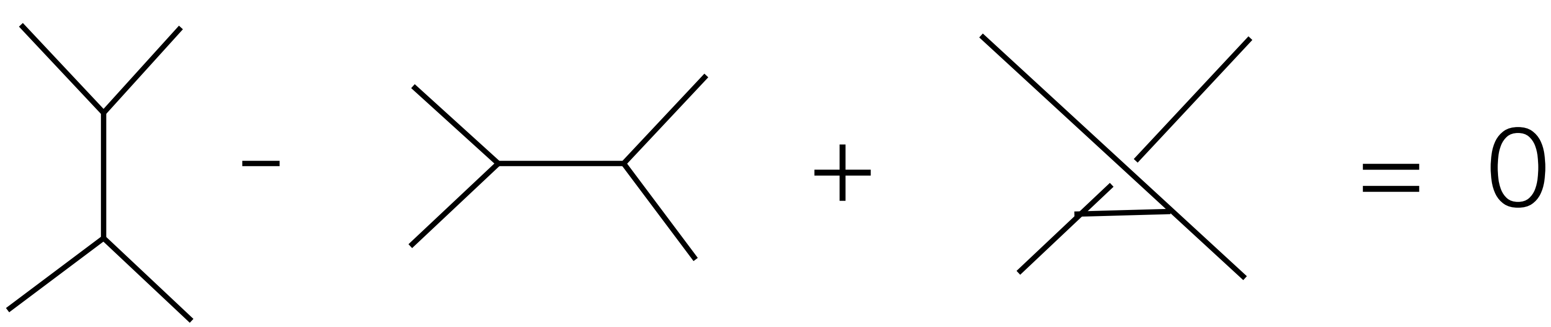
EXAMPLES:

(2) Put a planar trivalent graph in the black box



Merge by joining two leaves.

There are relations among the generators: IHX



Cyclic operads

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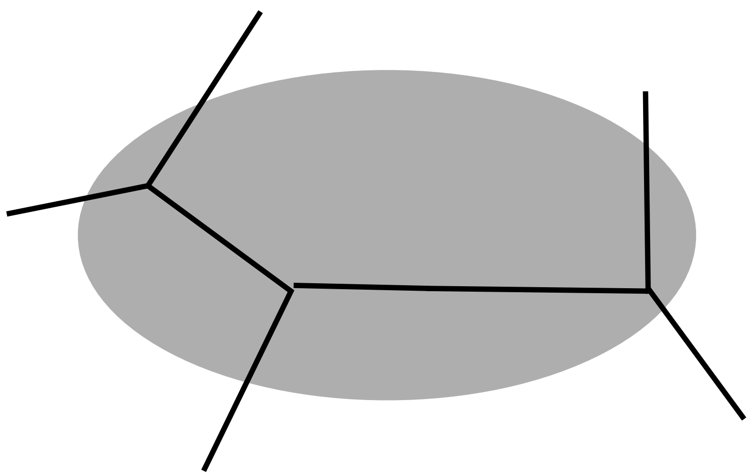

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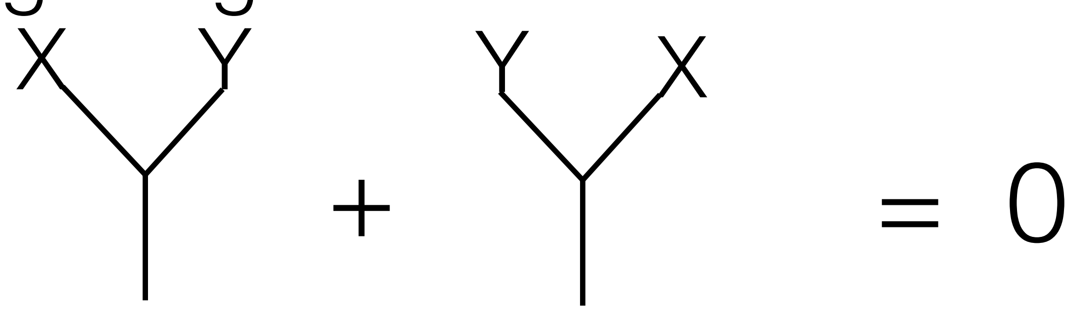


Lie algebra associated to $\mathcal{O}$

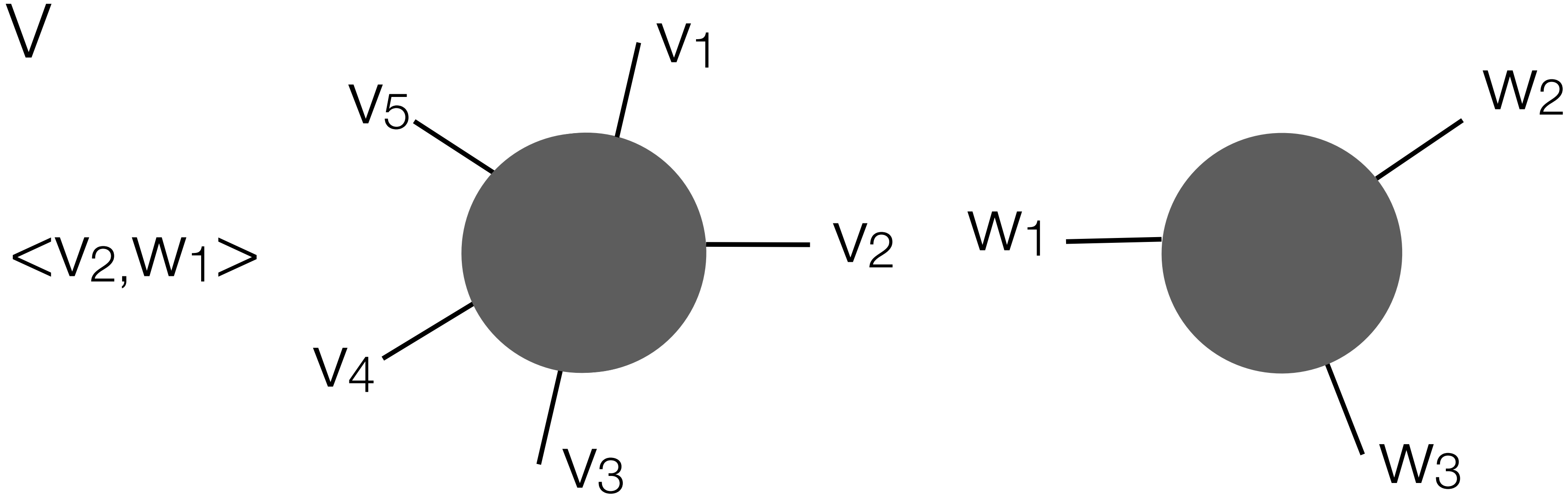
Given a cyclic operad $\mathcal{O}$ and a symplectic vector space V

Form a Lie algebra $\boldsymbol{\ell}_V$

Generators: $\mathcal{O}$-spiders

Bracket: Sum of all possible fusions

An $\mathcal{O}$-spider is an operad generator with i/o slots labelled by vectors in V



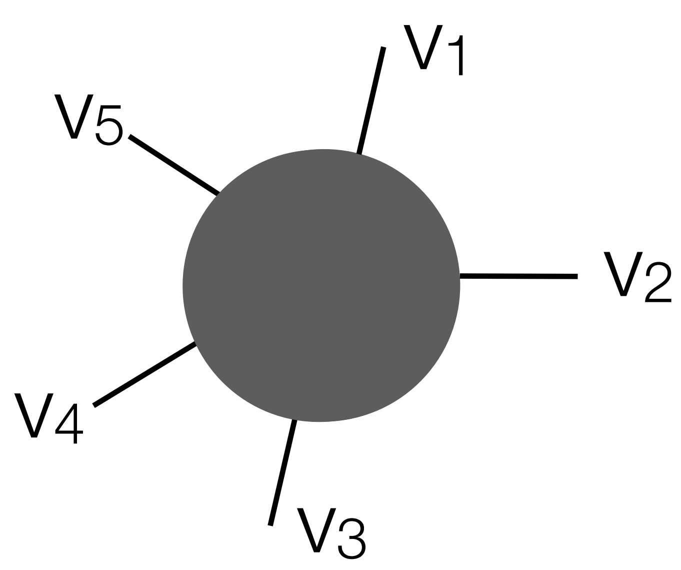
$\mathcal{O}$-spiders are fused by merging them using an i/o slot from each and multiplying by the symplectic product of their labels.

Symplectic action and Kontsevich's theorem

The symplectic group Sp=Sp(V) acts on ℓ_V.

Theorem [Kontsevich] For $\mathcal{O}$ = Lie,

$$PH_k(\ell_\infty)^{Sp} \cong \oplus_{r\geq 2} H^{2r-2-k}(Out(F_r))$$



Chain complex for $H_*(\ell_V)$

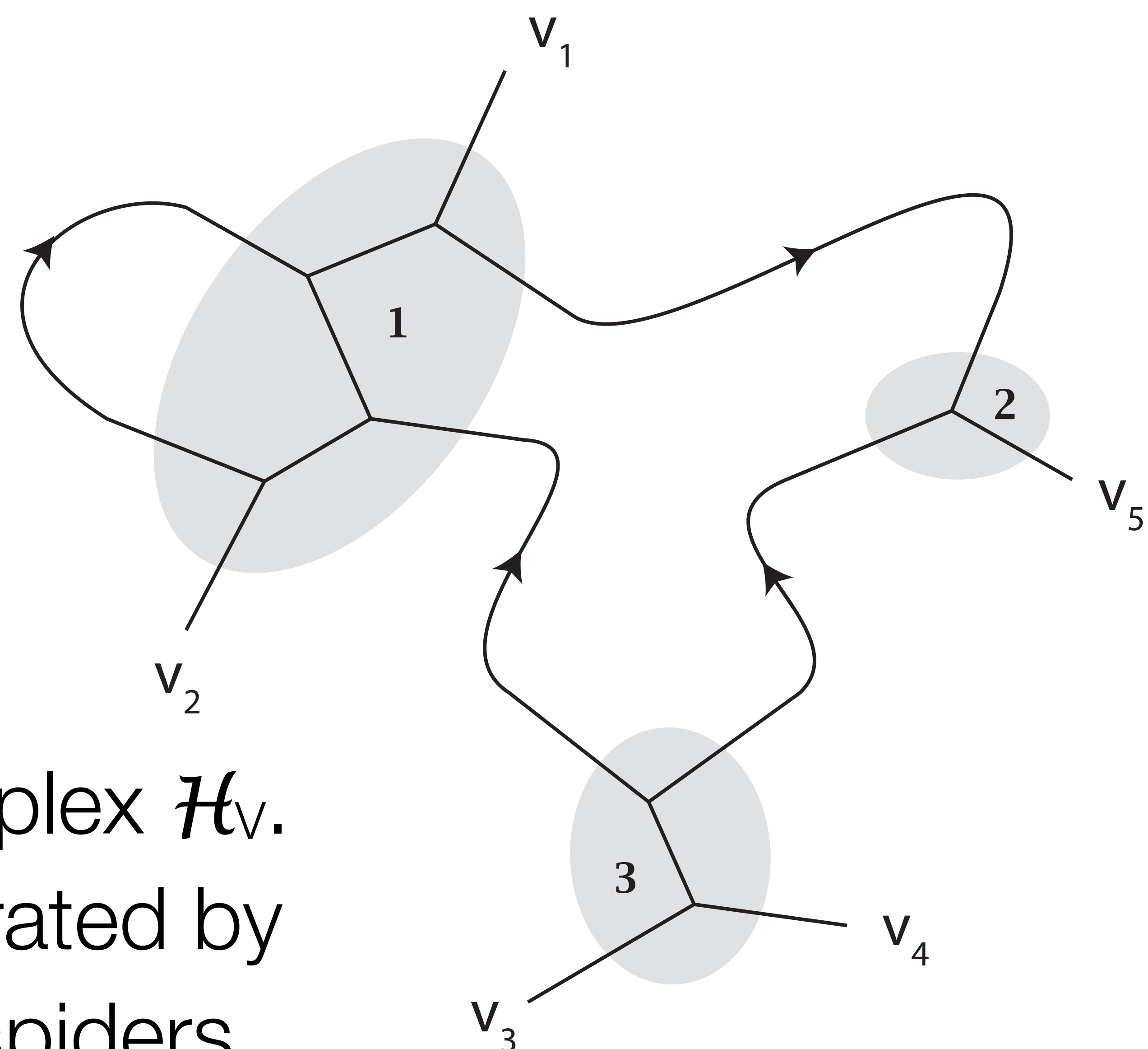
To compute the homology of ℓ_V use the exterior algebra with the Chevalley-Eilenberg differential:

$$\ldots \longrightarrow \wedge^k \ell_V \longrightarrow \wedge^{k-1} \ell_V \longrightarrow \ldots$$

$$\partial(x_1 \wedge \ldots \wedge x_k) = \sum_{i<j} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n$$

Hairy graphs and the trace map

A *hairy graph* is formed from an ordered list of spiders by erasing pairs of labels and joining the unlabeled legs by oriented edges:

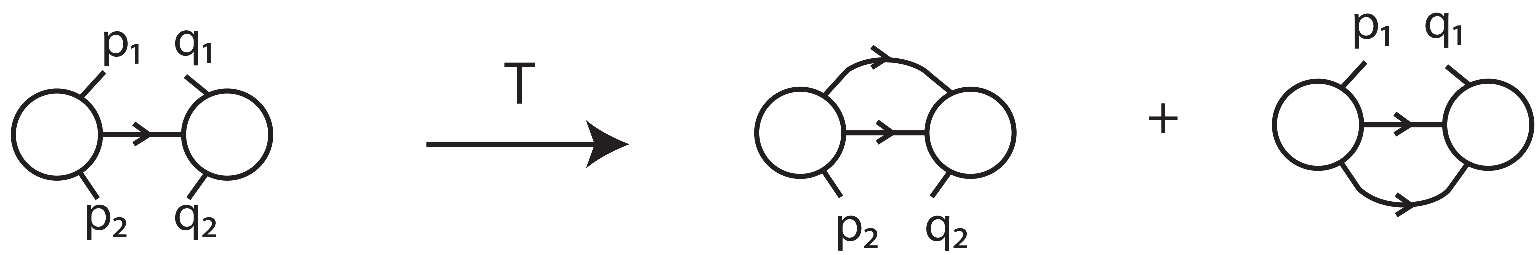


These generate a chain complex $\mathcal{H}_V$.
The k-chains $C_k\mathcal{H}_V$ are generated by hairy graphs formed from k spiders.
The boundary map $\partial_{\mathcal{H}}$ merges two spiders along an oriented edge in all possible ways.

Hairy graphs and the trace map

$i : \wedge \ell_V \to \mathcal{H}_V$ erases the wedge signs.
This is not a chain map.

$T : \mathcal{H}_V \to \mathcal{H}_V$ erases two labels, joins the unlabeled legs with an oriented edge, and multiplies by the symplectic product of the erased labels.



Proposition. The composition $Tr = e^T \circ i : \wedge \ell_V \to \mathcal{H}_V$ is a chain map.

Theorem. $Tr_*: H_i(\ell_V) \to H_i(\mathcal{H}_V)$ is injective.

Hairy graphs and the trace map

Theorem. $Tr_*: H_i(\ell_V) \to H_i(\mathcal{H}_V)$ is injective.

Theorem. $(p \circ Tr)_*: H_i(\ell_V) \to H_i(\mathcal{H}_{V^+})$ is surjective.

These theorems give upper and lower bounds on the size of $H_i(\ell_V)$ in terms of hairy graph homology.

Now: compute some hairy graph homology.

$H_1(\ell_\infty)$ for $\mathcal{O}$ = Assoc

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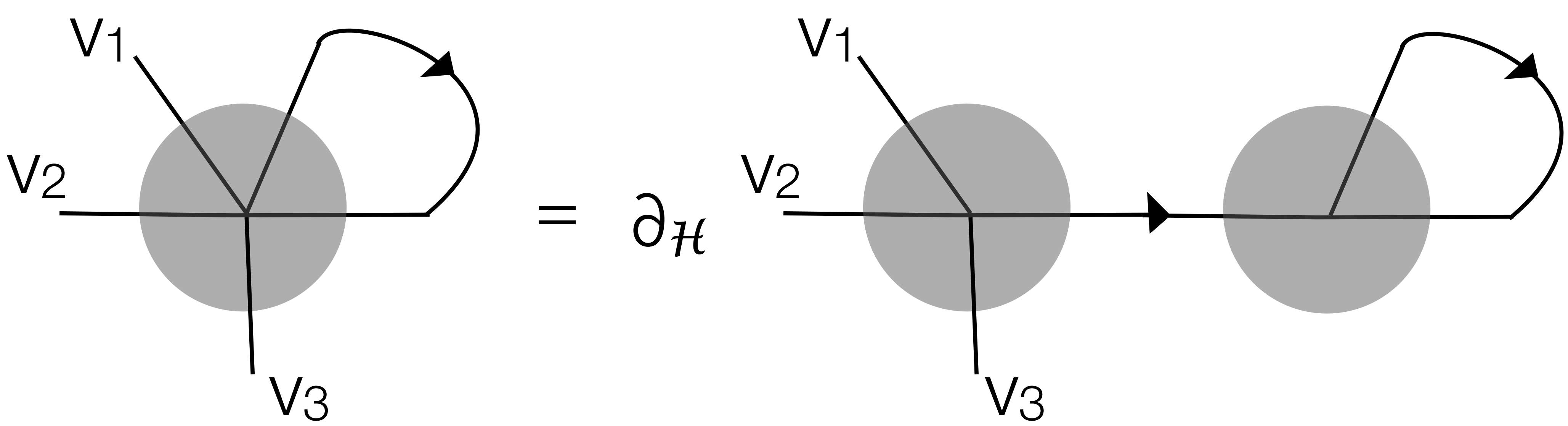
$$\ldots \to C_3\mathcal{H} \to C_2\mathcal{H} \to C_1\mathcal{H} \to 0$$

$$H_1(\mathcal{H}) = C_1\mathcal{H} / \partial_{\mathcal{H}} (C_2\mathcal{H})$$

Relations:



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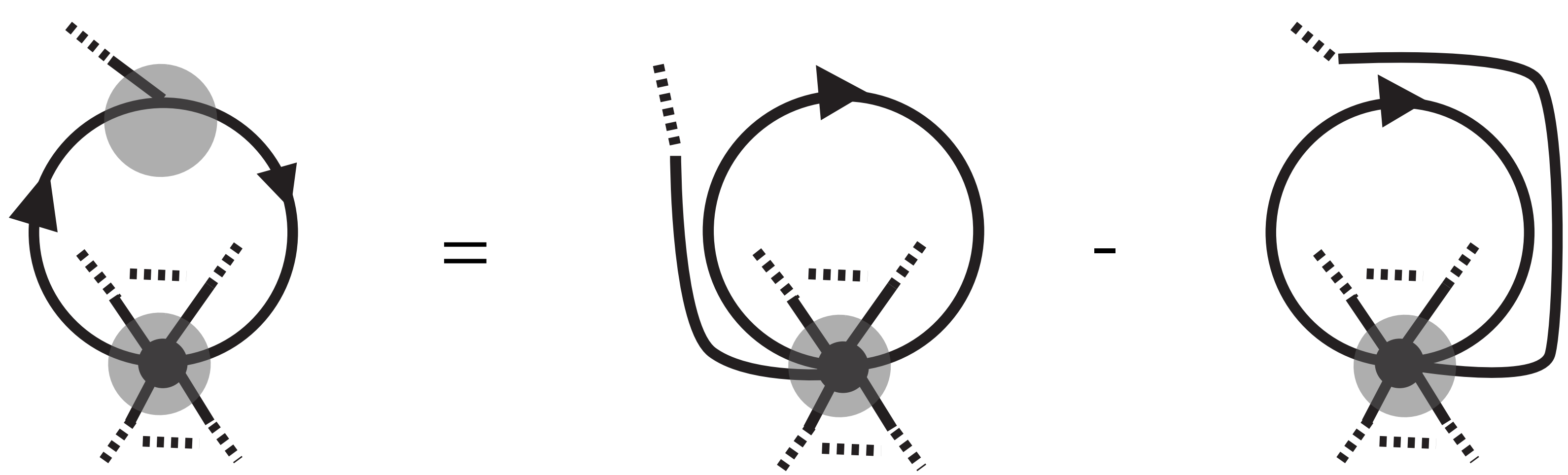
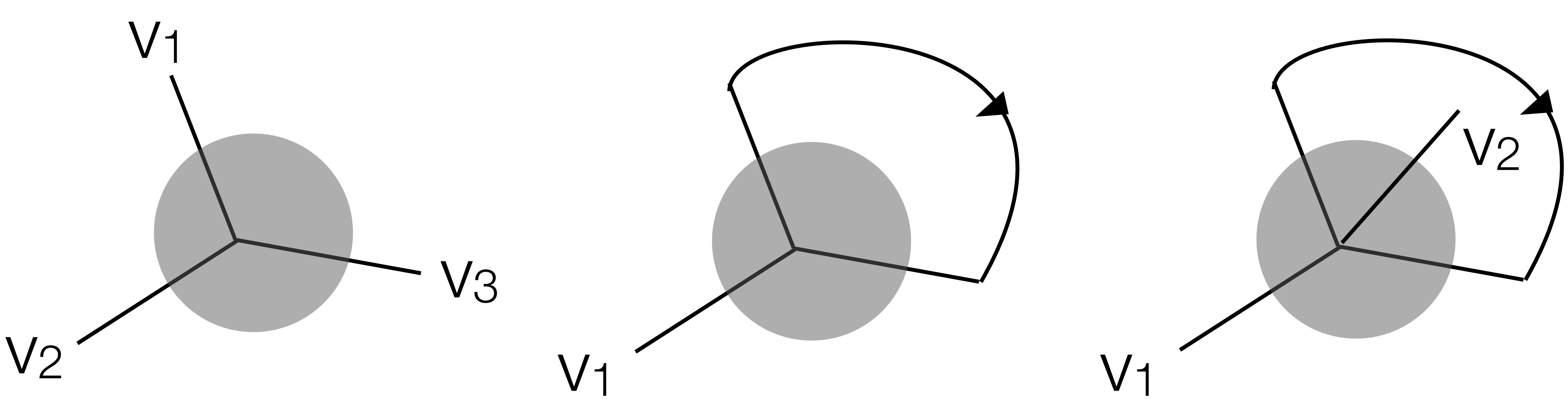
$\partial_{\mathcal{H}}$ 

$H_1(\ell_\infty)$ for $\mathcal{O}$ = Assoc

$$H_1(\mathcal{H}) = C_1\mathcal{H} / \partial_{\mathcal{H}} (C_2\mathcal{H})$$

Using "slide moves" and the boundary observation, can show:

Theorem: The only non-trivial generators of $H_1(\mathcal{H})$ are



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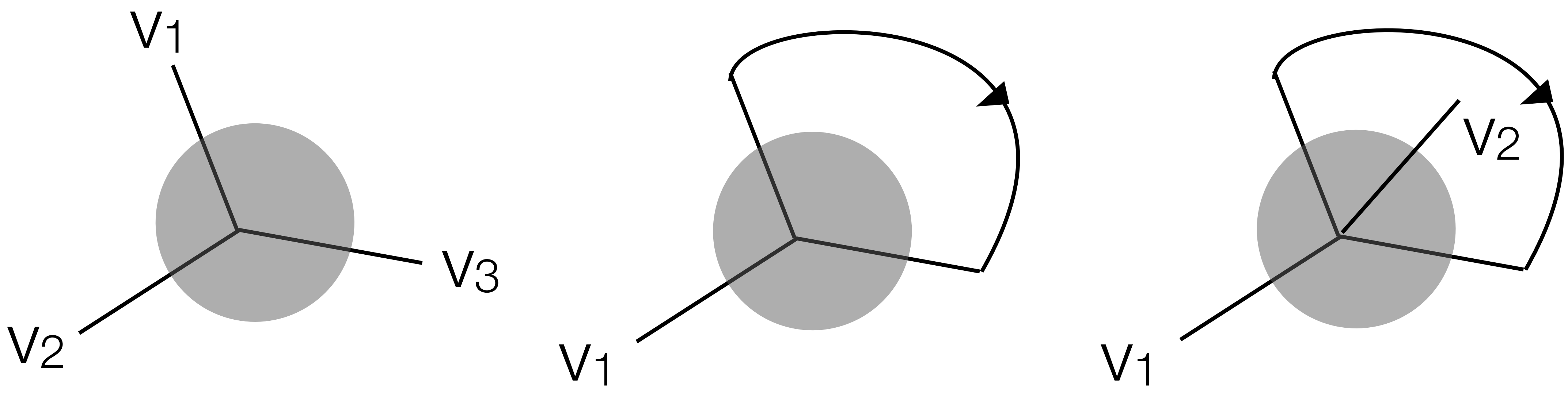


As GL(V) modules $H_1(\mathcal{H}) \cong [V^{\otimes 3}]_{\mathbb{Z}_3} \oplus V \oplus \wedge^2 V$

We can now compute the image of Tr in $H_1(\mathcal{H})$ to conclude

Corollary:

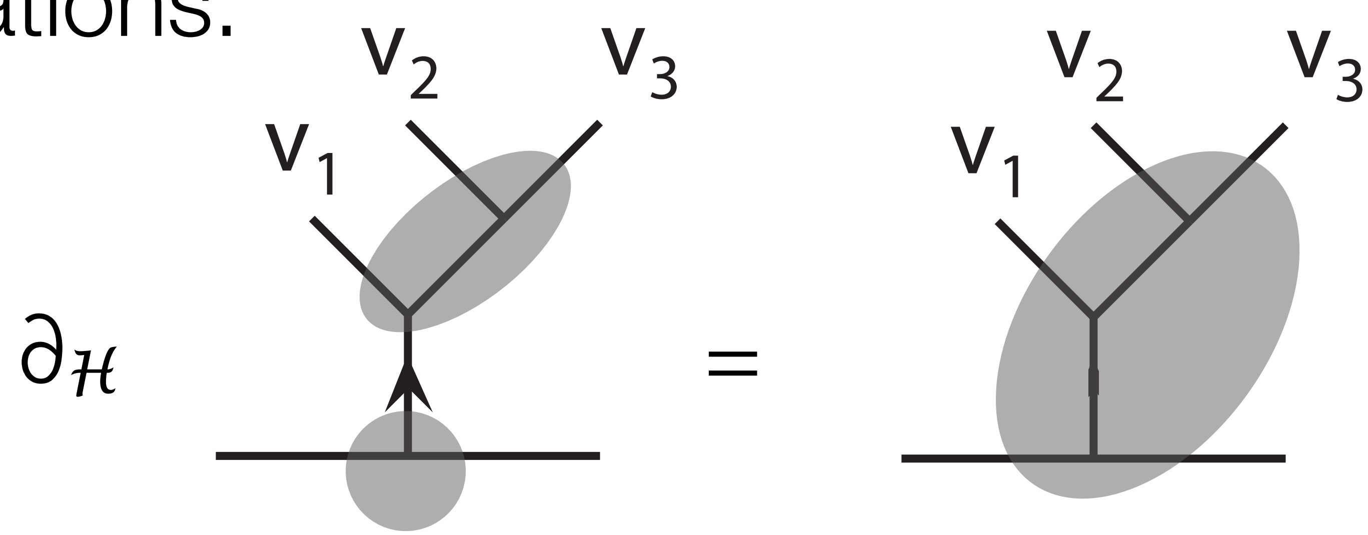
$$H_1(\ell_\infty) \cong [V^{\otimes 3}]_{\mathbb{Z}_3} \oplus (\wedge^2 V)/\langle \omega_0 \rangle \text{ where } \omega_0 = \sum_i p_i \wedge q_i$$

$H_1(\ell_\infty)$ for $\mathcal{O}$ = Lie

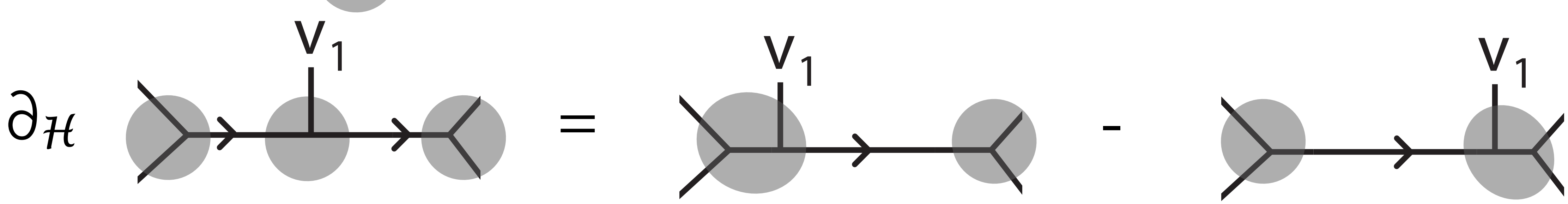
Again $H_1(\mathcal{H}) = C_1\mathcal{H} / \partial_{\mathcal{H}} (C_2\mathcal{H})$

Observations:

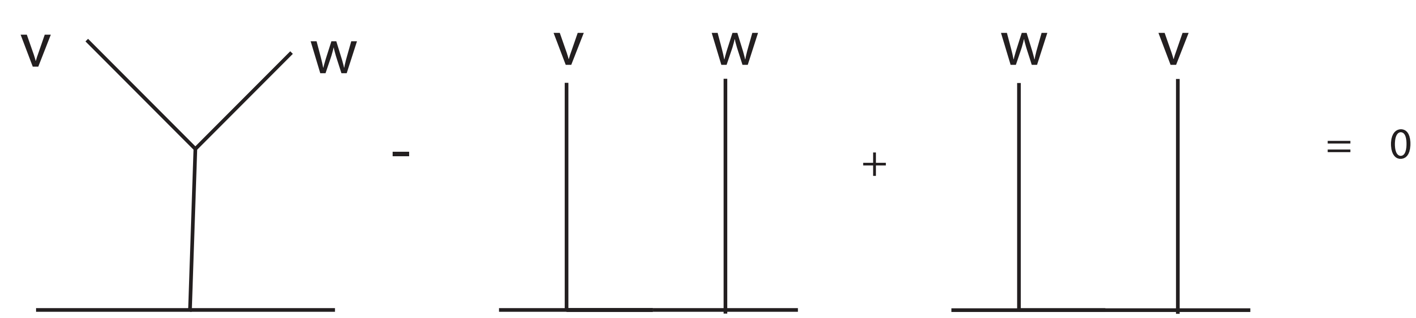
(1)



(2)



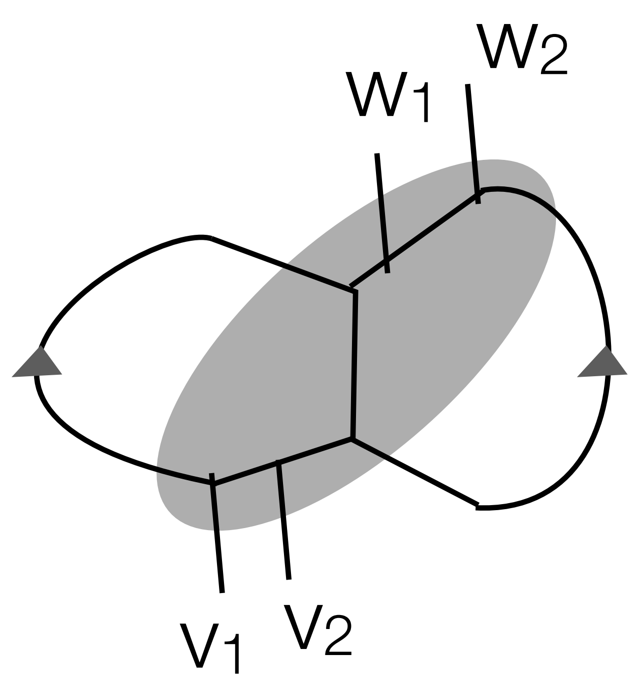
(3)



$H_1(\ell_\infty)$ for $\mathcal{O}$ = Lie

Again $H_1(\mathcal{H}) = C_1\mathcal{H} / \partial_{\mathcal{H}} (C_2\mathcal{H})$

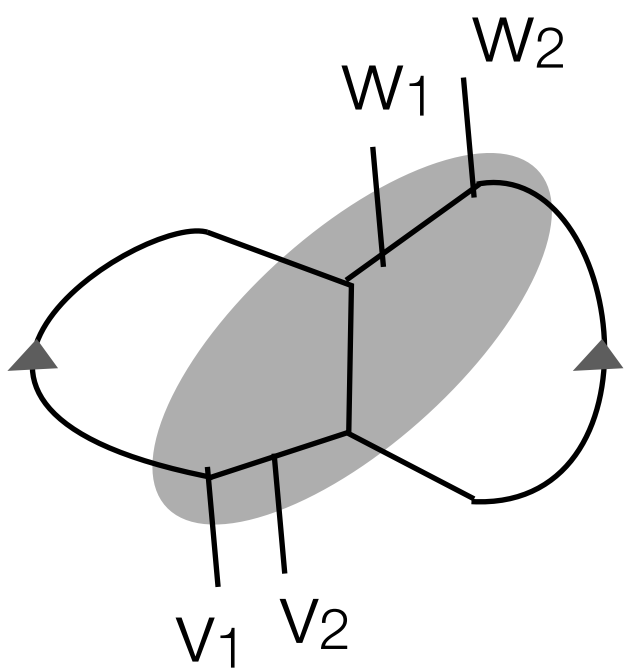
These imply $H_1(\mathcal{H})$ is generated by trivalent graphs with hairs on the oriented edges.



$v_1\ v_2\ w_1\ w_2$

Modulo $\partial_{\mathcal{H}}$ the order of the hairs does not matter, so each oriented edge gives a monomial in $\mathbb{C}[V]$

$H_1(\boldsymbol{\ell}_\infty)$ for $\mathcal{O}$ = Lie



$v_1\ v_2\ w_1\ w_2$

The graphs of rank r form a subcomplex of $\mathcal{H}$ so

$$H_k(\mathcal{H}) = \oplus_r \; H_{k,r}(\mathcal{H})$$

Theorem. $H_{1,0}(\mathcal{H}) \cong \wedge^3 V$

$$H_{1,1}(\mathcal{H}) \cong \oplus_k S^{2k+1} V$$

For r>1, $H_{1,r}(\mathcal{H}) \cong H^{2r-3}(Out(F_r); \mathbb{C}[V^r])$

$H_1(\ell_\infty)$ for $\mathcal{O}$ = Lie

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For r>1, $H_{1,r}(\mathcal{H}) \cong H^{2r-3}(Out(F_r); \mathbb{C}[V^r])$

In particular, $H_{1,2}(\mathcal{H}) \cong H^1(Out(F_2); \mathbb{C}[V^2])$

- $Out(F_2) \cong GL(2,\mathbb{Z})$
- $1 \to SL(2,\mathbb{Z}) \to GL(2,\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 1$
- $H^1(SL(2,\mathbb{Z}); \mathbb{C}[x,y])$ is computed in terms of modular forms by the Eichler-Shimura isomorphism.

Using these facts, we compute $H^1(Out(F_2); \mathbb{C}[V^2])$

$H_1(\ell_\infty)$ for $\mathcal{O}$ = Lie

Theorem. $H_{1,2}(\mathcal{H}) \cong H^1(Out(F_2); \mathbb{C}[V^2]) \cong \bigoplus_{k>l\geq 0} S_{(k,l)}V^{\oplus\lambda(k,l)}$

where

$S_{(k,l)}V$ is the irreducible GL(V) representatin associated to the partition (k,l)

$\lambda(k,l)$ is

the dimension of the space of weight k-l+2 modular forms if l is even

1+ the dimension of the space of weight k-l+2 modular forms if l is odd

$H_1(\ell_\infty)$ for $\mathcal{O}$ = Lie

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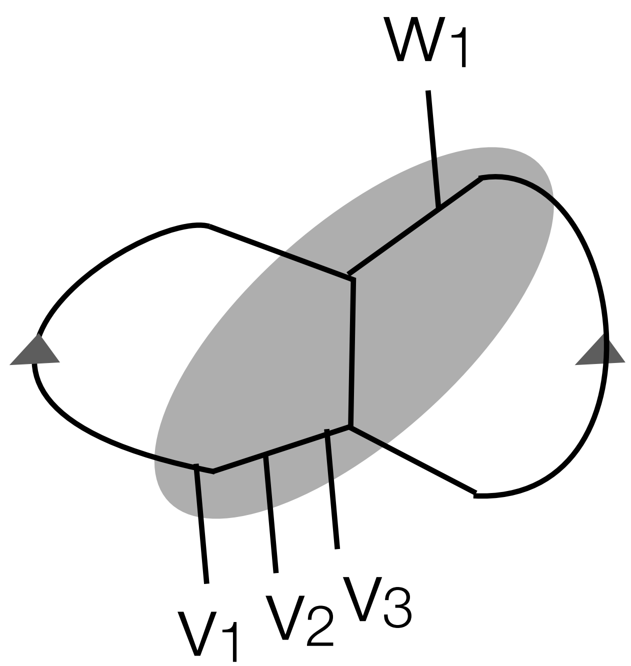
Example: A non-trivial element of $H_{1,2}(\mathcal{H})$:



with all symplectic products trivial

This is in the image of Tr, so represents a non-trivial element of $H_1(\ell_\infty)$

Constructing cohomology for Out(F_n)

We use elements of $H_k(\ell_\infty)$ to construct cohomology classes for Out(F_n), i.e. classes in $PH^*(\ell_\infty)^{Sp}$

Easiest to explain: abelianization $\ell_\infty \to H_1(\ell_\infty)$ induces a backwards map on Lie algebra cohomology

$$\mathfrak{g} \longrightarrow \mathfrak{a}$$

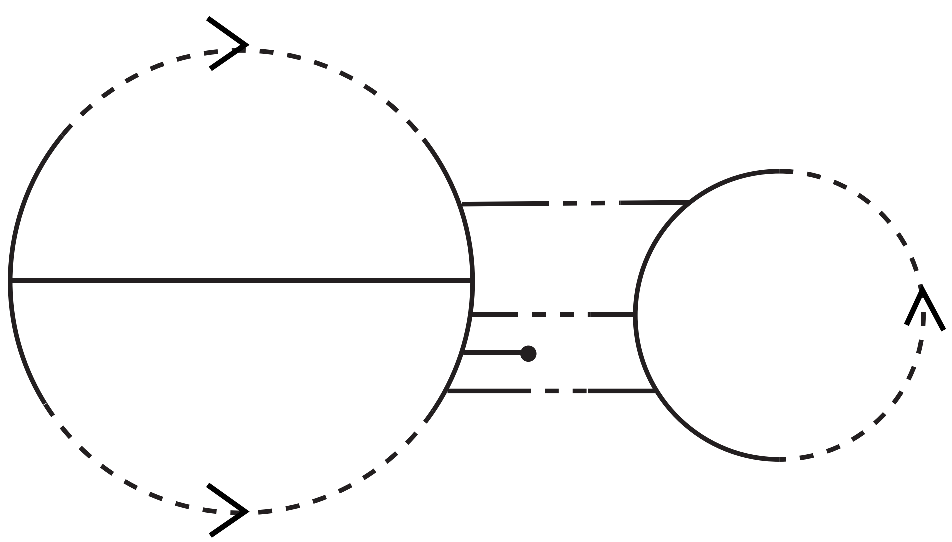
$$H^*(\mathfrak{a}) \longrightarrow H^*(\mathfrak{g})$$

$$\Lambda^*(\mathfrak{a}) \longrightarrow H^*(\mathfrak{g})$$

$$\Lambda^*(\mathfrak{a})^{\mathfrak{sp}} \longrightarrow H^*(\mathfrak{g})^{\mathfrak{sp}}$$

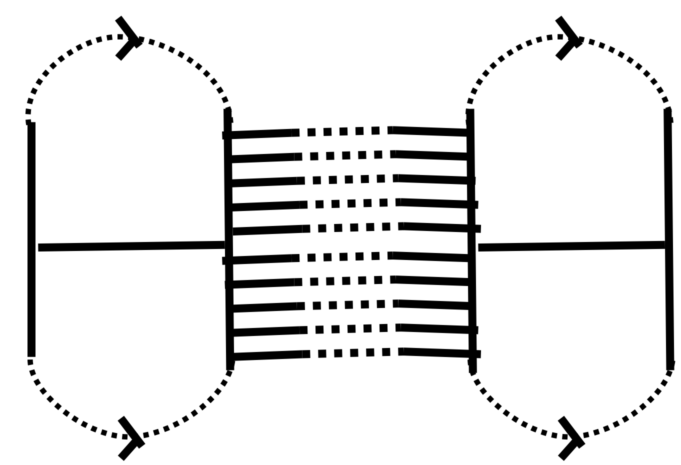
Constructing cohomology for Out(F_n)

We can use elements of $H_k(\ell_\infty)$ to construct cohomology classes for Out(F_n), i.e. classes in $PH^*(\ell_\infty)^{Sp}$



$H_7(Aut(F_5);\mathbb{Q})$

$H_{11}(Aut(F_7);\mathbb{Q})$



$H_{22}(Out(F_{13});\mathbb{Q})$