Cut-points in asymptotic cones of groups

Mark Sapir

With J. Behrstock, C. Druțu, S. Mozes, A.Olshanskii, D. Osin

Asymptotic cones

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- The a.c. of $\mathbb{Z}^2$ is $\mathbb{R}^2$,
- the a.s. of a binary tree is an $\mathbb{R}$-tree

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Observation due to Bestvina and Paulin: if a group has many actions on a Gromov-hyperbolic metric space then it acts non-trivially (i.e. without a global fixed point) by isometries on the asymptotic cone of that space which is an $\mathbb{R}$-tree.

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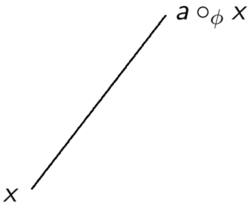

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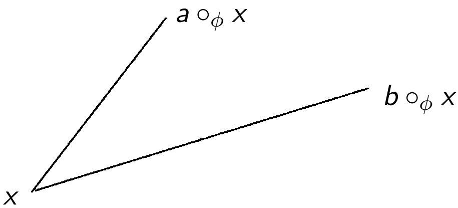

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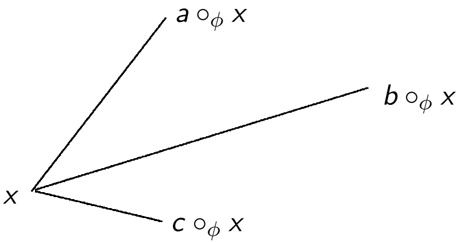

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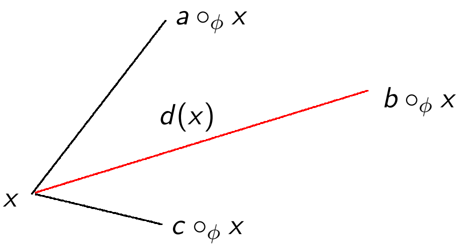

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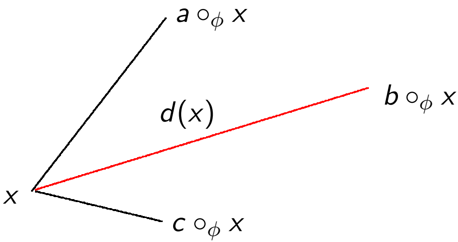

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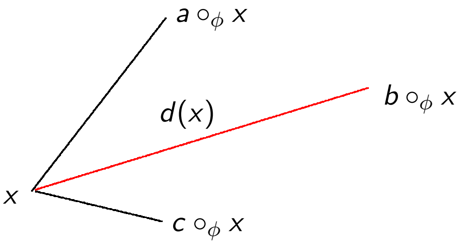

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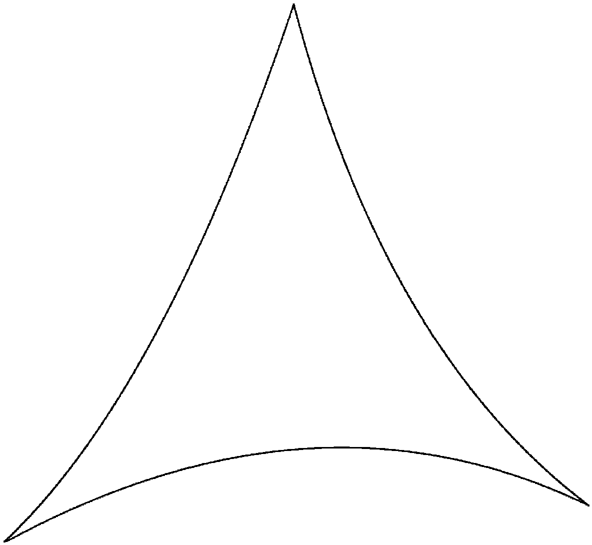
Then we can divide the metric in X by d_ϕ, obtaining X_ϕ, $\phi \colon \Lambda \to G$. The $\mathbb{R}$-tree is the limit $\mathrm{Con}(X, (d_\phi), (x_\phi))$.

Indeed, triangles in X_ϕ become thinner and thinner and in the limit all geodesic triangles are tripods:



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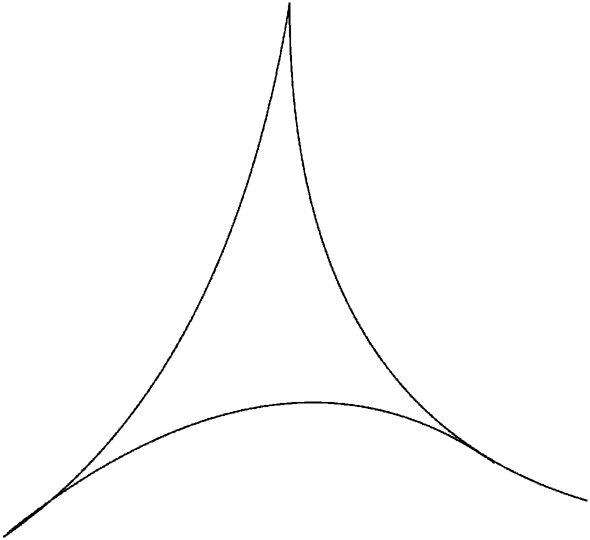

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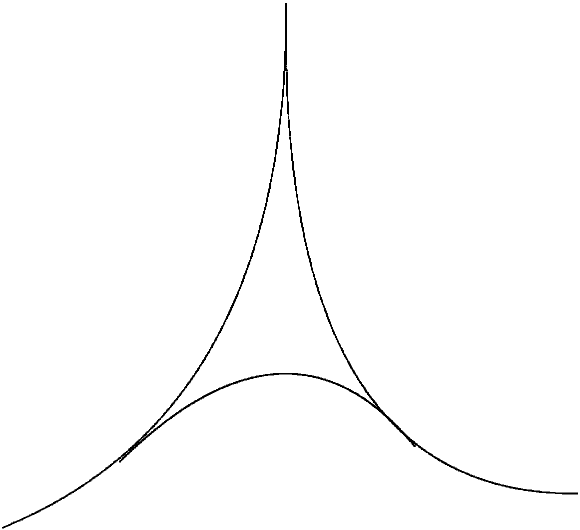

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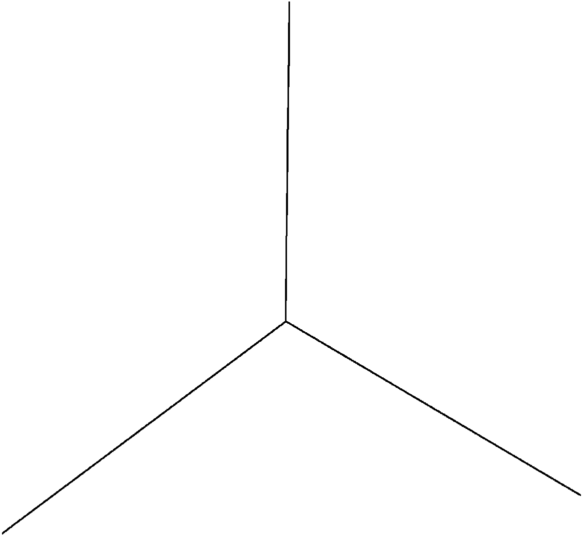

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Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$.

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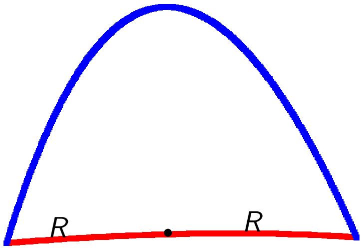

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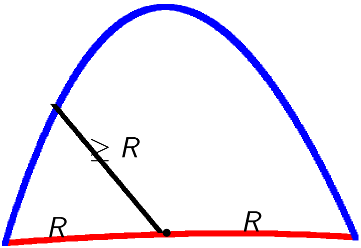

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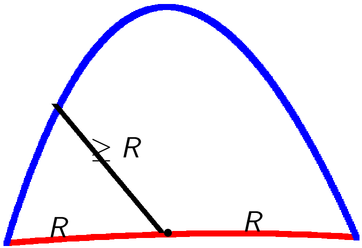

The length of the blue arc should be $> O(R)$.

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Cut points and tree-graded structures

Recall that hyperbolicity $\equiv$superlinear divergence of any pair of geodesic rays with common origin.

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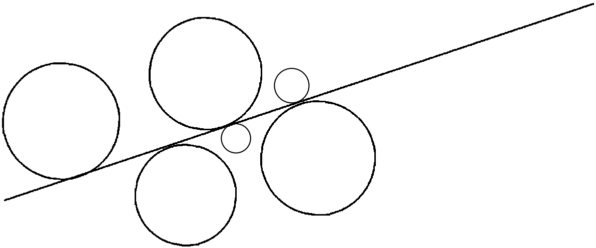

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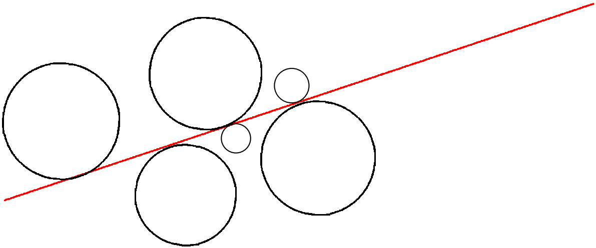

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The line is a transversal tree, the other transversal trees are points on the circles.

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Obvious Lemma (Druțu, Mozes, S.)

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Observation. (Druțu+S.) A bi-infinite geodesic in the Cayley graph is Morse iff its limit in every asymptotic cone is a transversal geodesic.

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Obvious Lemma (Druțu, Mozes, S.) If $H < G$, $g \in H$ is such that $\{g^n, n \in \mathbb{Z}\}$ is Morse in G (i.e. g is a Morse element). Then g is Morse in H.

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Corollary. If H does not have cut-points in its asymptotic cones (say, H is a lattice in $\mathrm{S}L_n(\mathbb{R})$ by DMS or satisfies a law by DS) then every injective image of H in a MCG does not contain pseudo-Anosov elements and hence is reducible.

Actions on tree-graded spaces

The main property of tree-graded spaces: if a geodesic $\mathfrak{p}$ connecting $u = \mathfrak{p}_-$ and $\mathfrak{p}_+ = v$ enters a piece in point a and exits in point $b \neq a$, then every path connecting u and v passes through a and b. Thus for every pair of points u, v we can define $\tilde{d}(a, b)$ as $\mathrm{dist}(a, b)$ minus the sum of lengths of subgeodesics which are inside pieces. We have that $\tilde{d}$ is a pseudo-distance. Let $\sim$ be the equivalence relation $a \sim b$ iff $\tilde{d}(a, b) = 0$.

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Theorem (D+S) $X/\sim$ is an $\mathbb{R}$-tree.

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Question. Is there a f.g. (f.p.) amenable group with cut points in every a.c.?

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3. Every two universal $\mathcal{Q}$-trees are isometric.

Relatively hyperbolic groups

Theorem. (Osin+S) Let G be a group generated by a finite set X and hyperbolic relative to a collection of subgroups $\{H_1, \dots, H_n\}$. Then for every non-principal ultrafilter ω and every scaling sequence $d = (d_i)$, the asymptotic cone of G is bi-Lipschitz equivalent to the universal $\mathcal{Q}$-tree, where $\mathcal{Q} = \{\mathrm{Con}^{\omega}(H_i, d) \mid i = 1, \dots, n\}$.

Other groups with tree-graded asymptotic cones.

Let G be the fundamental group of a hyperbolic knot complement. Then it is hyperbolic relative to a free abelian subgroup of rank 2 and all asymptotic cones of G are bi-Lipschitz equivalent to the universal $\{\mathbb{R}^2\}$-tree. The same holds, for asymptotic cones of $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$.

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Similarly, every non-uniform lattice in $\mathrm{SO}(n, 1)$ is relatively hyperbolic with respect to finitely generated free Abelian subgroups $\mathbb{Z}^{n-1}$, hence their asymptotic cones are all bi-Lipschitz equivalent to the asymptotic cones of $\mathbb{Z}^{n-1} * \mathbb{Z}$ and are bi-Lipschitz equivalent to the universal $\{\mathbb{R}^{n-1}\}$-tree.

Theorem (Osin+ S)[Assuming CH is true] Let $\mathbb{F}$ be an asymptotic cone of a geodesic metric space. Suppose that $\mathbb{F}$ is homogeneous and has cut points. Then $\mathbb{F}$ is isometric to the universal $\mathcal{Q}$-tree, where $\mathcal{Q}$ consists of representatives of isometry classes of maximal connected subspaces of $\mathbb{F}$ without cut points.

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Thus, modulo CH, the asymptotic cones of the mapping class groups are completely determined by their pieces, that is maximal connected subsets without cut points. These have been described in [Behrstock-Kleiner-Minsky-Mosher] and [Behrstock-Druțu-S].

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Application to MCG

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This is based on the following

Proposition.[Bestvina, Bromberg, Fujiwara] There exists an explicitly defined finite index torsion-free subgroup $\mathcal{BBF}(S)$ of $\mathcal{MCG}(S)$ such that the set of all subsurfaces of S can be partitioned into a finite number of subsets $C_1, C_2, ..., C_s$, each of which is an orbit of $\mathcal{BBF}(S)$, and any two subsurfaces in the same subset overlap and have the same complexity.