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ESTIMATES OF TOPOLOGICAL COMPLEXITY

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ABSTRACT

The topological complexity $\mathrm{TC}(X)$ of a path connected space X is a homotopy invariant introduced by M. Farber in 2003 in his work on motion planning in robotics. $\mathrm{TC}(X)$ reflects the complexity of the problem of choosing a path in a space X so that the choice depends continuously on its endpoints. More precisely $\mathrm{TC}(X)$ is defined to to be the minimal integer n for which $X\times X$ admits an open cover $U_1,\ldots,U_n$ such that the fibration $(\mathrm{ev}_0,\mathrm{ev}_1)\colon X^I\to X\times X$ admits local sections over each U_i . This is reminiscent of the definition of $\mathrm{LS}(X)$ the Lusternik-Schnirelmann category of the space, and in fact the two concepts can be seen as special cases of the so-called Schwarz genus of a fibration. In a somewhat different vein Iwase and Sakai (2008) observed that the topological complexity can be seen as a fibrewise Lusternik-Schnirelmann category. Both invariants are notoriously difficult to compute, so we normally rely on the computation of various lower and upper estimates. In this talk we use the Iwase-Sakai approach to discuss some of these estimates and their relations.

This is joint work with Aleksandra Franc

TOPOLOGICAL COMPLEXITY

X path-connected

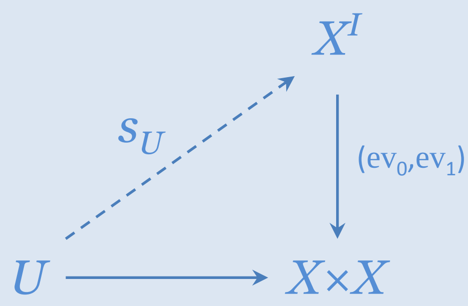
Motion plan for X is a map that to every pair of points $(x_0,x_1)\in X\times X$ assigns a path $\alpha:(I,0,1)\to(X, x_0,x_1)$. In fact, such a plan exists if, and only if X is contractible.

Local motion plan over $U\subseteq X\times X$ is a map that to every pair of points $(x_0,x_1)\in U$ assigns a path $\alpha:(I,0,1)\to(X, x_0,x_1)$.

(Farber 2003) Topological complexity of X, $\mathrm{TC}(X)$, is the minimal number of local motion plans needed to cover $X\times X$.

TOPOLOGICAL COMPLEXITY

A local motion plan over U is a local section of the evaluation fibration



$\mathrm{TC}(X) = \mathrm{secat}((\mathrm{ev}_0,\mathrm{ev}_1)\colon X^I \to X\times X)$

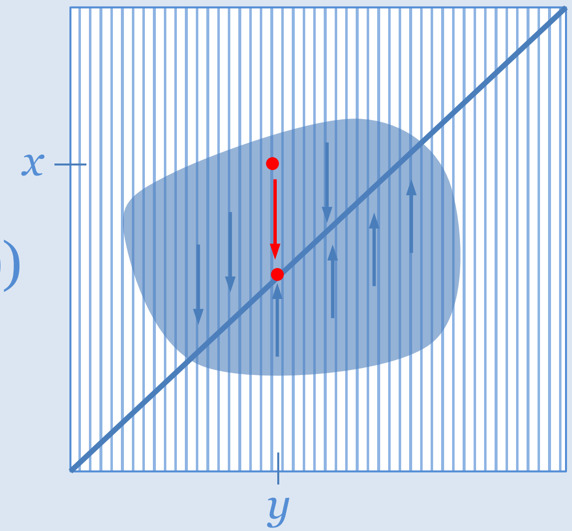
(<u>sectional category</u> = minimal n, such that $X\times X$ can be covered by n open sets that admit local sections)

(also called <u>Švarc genus</u> of the evaluation fibration)

IWASE – SAKAI REFORMULATION

Local section $s_U\colon U \to X^I$ corresponds to a vertical deformation of U to the diagonal $\Delta \subset X \times X$.

$$H : U \times I \to X \times X, \;\; (x, y, t) \mapsto (y, s_U(x, y)(t))$$



Iwase-Sakai (2010):

$$\mathrm{TC}(X) = \text{fibcat}\begin{pmatrix} X \times X \\ \mathrm{pr}_1 \downarrow \; \uparrow \Delta \\ X \end{pmatrix}$$

fibrewise (pointed) category = minimal n, such that $X \times X$ can be covered by n open sets that admit vertical deformation to the diagonal

Gives more geometric approach. On each fibre get a categorical cover of X.

Topological complexity is fibrewise LS-category.

WHITEHEAD-TYPE CHARACTERIZATION OF TOPOLOGICAL COMPLEXITY

For a pointed construction $X \mapsto CX$, define $X \rtimes CX$ to be the fibrewise space over X with base point determined by the first coordinate.

Example: $X \rtimes \mathrm{W}^n X = \{(x, x_1, \ldots, x_n);\ x_i = x \text{ for some } i\}$ (fibrewise fat wedge)

$$\mathrm{TC}(X) \le n \iff \begin{array}{ccc} & & X \rtimes \mathrm{W}^n X \\ & \overset{s}{\nearrow} & \big\downarrow {\scriptstyle 1 \rtimes i_n} \\ X \rtimes X & \xrightarrow[1 \rtimes \Delta_n]{} & X \rtimes \Pi^n X \end{array}$$

$\exists s:\ 1 \rtimes i_n \circ s$ is vertically homotopic to $1 \rtimes \Delta_n$

Proof: (assume X normal, all points non'degenerate) Deformations of U_i to the diagonal determine a deformation of the fibrewise product to the fibrewise fat wedge.

GANEA-TYPE CHARACTERIZATION OF TOPOLOGICAL COMPLEXITY

Ganea construction: start with $\mathrm{G}^0X=\mathrm{P}X$ (based paths) and $p_0\colon \mathrm{P}X \to X$, and inductively define $\mathrm{G}^{n+1}X := \mathrm{G}^nX \cup \mathrm{cone}(\text{fibre of } p_n)$.

This is also a pointed construction so we get $1 \rtimes p_n\colon X \rtimes \mathrm{G}^nX \to X \rtimes X$.

$$\mathrm{TC}(X) \leq n \iff 1 \rtimes p_n\colon X \rtimes \mathrm{G}^nX \to X \rtimes X \text{ admits a section.}$$

Proof: Show

$$\begin{array}{ccc} X \rtimes \mathrm{G}^nX & \longrightarrow & X \rtimes \mathrm{W}^nX \\ {\scriptstyle 1 \rtimes p_n}\downarrow & & \downarrow{\scriptstyle 1 \rtimes i_n} \\ X \rtimes X & \xrightarrow[1 \rtimes \Delta_n]{} & X \rtimes \Pi^nX \end{array}$$

is the homotopy pullback.

LOWER BOUNDS FOR TC

We can summarize the relations in a diagram of spaces over X:

$$
\begin{array}{ccc}
X \rtimes G^n X & \longrightarrow & X \rtimes W^n X \\
\downarrow{\scriptstyle 1\rtimes p_n} & & \downarrow{\scriptstyle 1\rtimes i_n} \\
X \rtimes X & \xrightarrow[1\rtimes \Delta_n]{} & X \rtimes \Pi^n X \\
\downarrow{\scriptstyle 1\rtimes q'_n} & & \downarrow{\scriptstyle 1\rtimes q_n} \\
X \rtimes G^{[n]} X & \longrightarrow & X \rtimes \Lambda^n X
\end{array}
$$

$\mathrm{TC}(X) \leq n \iff 1\rtimes p_n$ admits a section

$\iff 1\rtimes \Delta_n$ lifts vertically along $1\rtimes i_n$

By analogy with the Lusternik-Schnirelmann category define:

$\mathrm{w'TC}(X)$:=min{ n; $1\rtimes q'_n \simeq_X$ section}

$\mathrm{wTC}(X)$:=min{ n; $(1\rtimes q_n)(1\rtimes \Delta_n) \simeq_X$ section}

$\mathrm{cTC}(X)$:=min{ n; $\Sigma_X(1\rtimes q_n)(1\rtimes \Delta_n) \simeq_X$ section}

$$\mathrm{TC}(X) \geq \mathrm{w'TC}(X) \geq \mathrm{wTC}(X) \geq \mathrm{cTC}(X) \geq \text{nil } H^*(X\times X, \Delta(X))$$

Conjecture: all inequalities can be strict.

LOWER BOUNDS FOR TC

Similarly

$$
\begin{array}{ccc}
X \rtimes G^n X & \longrightarrow & X \rtimes W^n X \\
\downarrow {\scriptstyle 1 \rtimes p_n} & & \downarrow {\scriptstyle 1 \rtimes i_n} \\
X \rtimes X & \xrightarrow[1 \rtimes \Delta_n]{} & X \rtimes \Pi^n X \\
\downarrow {\scriptstyle 1 \rtimes q'_n} & & \downarrow {\scriptstyle 1 \rtimes q_n} \\
X \rtimes G^{[n]} X & \longrightarrow & X \rtimes \Lambda^n X
\end{array}
$$

$\sigma TC(X)$:=min{ n; some $(\Sigma_X)^i(1 \rtimes q'_n) \simeq_X$ section }

$eTC(X)$:=min{ n; $(1 \rtimes p_n): H_*(X \rtimes G^n X, X) \rightarrow H_*(X \times X, \Delta(X))$ is epi }

nil $H^*(X \times X, \Delta(X)) \leq eTC(X) \leq \sigma TC(X) \leq w'TC(X) \leq TC(X)$