Poincaré inequalities and rigidity for actions on Banach spaces

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Property (T)

Property (T) was defined by Kazhdan in late 1960'ies.

We use a characterization of (T) due to Delorme – Guichardet as a definition.

Definition

A group G has Kazhdan's property (T) if every action of G by affine isometries on a Hilbert space has a fixed point.
Equivalently,

$$H^1(G,\pi) = 0$$

for every unitary representation π.

Generalizing (T) to other Banach spaces

X – Banach space, reflexive ($X^{**} = X$)

Example: L_p are reflexive for $1 < p < \infty$, not reflexive for $p = 1, \infty$.

We are interested in groups G for which the following property holds:

every affine isometric action of G on X has a fixed point

or equivalently,

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This is much more more difficult than for L_2, even when $X = L_p$.

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Previous results

Only a few positive results are known:

- (T) $\Longleftrightarrow$ fixed points on L_p and any subspace, $1 < p \leq 2$
- (T) $\Longrightarrow\ \exists\, \varepsilon = \varepsilon(G)$ such that fixed points always exists on L_p for $p \in [2, 2+\varepsilon)$ (Fisher – Margulis 2005)
(a general argument, ε unknown)
- lattices in products of higher rank simple Lie groups for $X = L_p$ for all $p > 1$ (Bader – Furman –Gelander – Monod, 2007)
- $\mathrm{SL}_n(\mathbb{Z}[x_1, \ldots x_k])$ for $n \geq 4$; $X = L_p$ for all $p > 1$ (Mimura, 2010)

  [both use a representation-theoretic Howe-Moore property]
- Gromov's random groups containing expanders for $X = L_p$, p-uniformly convex Banach lattices for all $p > 1$ (Naor – Silberman, 2010)

[Some of these arguments also apply to Shatten p-class operators]

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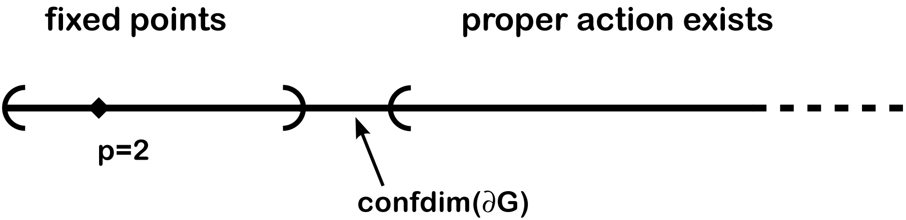
Previous results

Some groups with property (T) admit fixed point free actions on certain L_p.

- $Sp(n,1)$ admits fixed point free actions on $L_p(G)$, $p \geq 4n+2$ (Pansu 1995)
- hyperbolic groups admit fixed point free actions on $\ell_p(G)$ for $p \geq 2$ sufficiently large (Bourdon and Pajot, 2003)
- for every hyperbolic group G there is a $p > 2$ (sufficiently large) such that G admits a metrically proper action by affine isometries on $\ell_p(G \times G)$ (Yu, 2006)

Values of p (after C. Drutu)

Consider e.g. a hyperbolic group G with property (T).



There are many natural questions about the above values of p.

Let $\mathscr{P} = \left\{p : H^1(G,\pi) = 0 \text{ for every isometric rep. } \pi \text{ on } L_p\right\}$

The only thing we know about $\mathscr{P}$ is that it is open.

Question: Is $\mathscr{P}$ connected?

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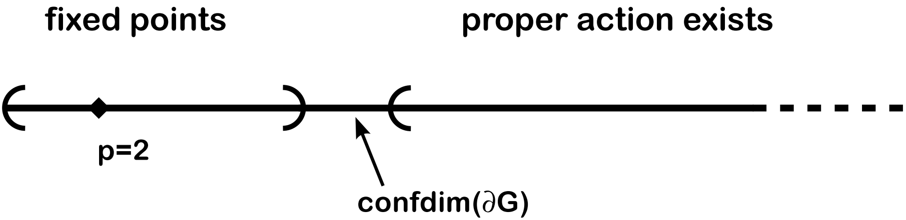

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Spectral conditions for property (T)

Based on the work of Garland,

used by Ballmann – Świątkowski, Dymara – Januszkiewicz, Pansu, Żuk

...

Theorem (General form of the theorems)

Let G be acting properly discontinuously and cocompactly on a 2-dimensional contractible simplicial complex K and denote by $\lambda_1(x)$ the smallest positive eigenvalue of the discrete Laplacian on the link of a vertex $x \in K$. If

$$\lambda_1(x) > \frac{1}{2}$$

for every vertex $x \in K$ then G has property (T).

Link graphs on generating sets

G - group, $S = S^{-1}$ - finite generating set of G, $e \notin S$.

Definition

The link graph $\mathcal{L}(S) = (V, E)$ of S:

- vertices $V = S$,
- $(s, t) \in S \times S$ is an edge $\in E$ if $s^{-1}t \in S$.

Laplacian on $\ell_2(S, \deg)$:

$$\Delta f(s) = f(s) - \frac{1}{\deg(s)} \sum_{t \sim s} f(t)$$

λ_1 denotes the smallest positive eigenvalue

Theorem (Żuk)

If $\mathcal{L}(S)$ connected and $\lambda_1(\mathcal{L}(S)) > \frac{1}{2}$ then G has property (T).

Poincaré inequalities

Let $Mf = \sum_{x \in V} f(x) \dfrac{\deg(x)}{\#E}$ be the mean value of f

Definition (p-Poincaré inequality for the norm of X)

X-Banach space, $p \geq 1$, $\Gamma = (V, E)$ - finite graph. For every $f : V \to X$

$$\left(\sum_{s \in V} \|f(s) - Mf\|_X^p \deg(s) \right)^{1/p} \leq \kappa \left(\sum_{(s,t) \in E} \|f(s) - f(t)\|_X^p \right)^{1/p}.$$

The inf of κ for $\mathcal{L}(S)$, giving the optimal constant, is denoted $\kappa_p(S, X)$

The classical p-Poincaré inequality when $X = \mathbb{R}$.

1. $\kappa_1(S, \mathbb{R}) \simeq$ Cheeger isoperimetric const
2. $\kappa_2(S, \mathbb{R}) = \sqrt{\lambda_1^{-1}}$;
3. for $1 \leq p < \infty$ we have $\kappa_p(S, L_p) = \kappa_p(S, \mathbb{R})$

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The Main Theorem

Given $p > 1$ denote by p^* the adjoint index: $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$.

Main Theorem

Let X be a reflexive Banach space, G a group generated by S as earlier. If for some $p > 1$

$$\max\left\{ 2^{-\frac{1}{p}} \kappa_p(S,X),\ 2^{-\frac{1}{p^*}} \kappa_{p^*}(S,X^*) \right\} < 1$$

then

$$H^1(G,\pi) = 0$$

for any isometric representation π of G on X.

Remark 1. By reflexivity, the same conclusion holds for actions on X^*

Remark 2. The roles of the two constants in the proof are different.

Sketch of proof

Difficulty: lack of self-duality when X is not a Hilbert space

For any Hilbert space $\mathcal{H}^* = \mathcal{H}$, every subspace has an orthogonal complement

For $Y \subseteq X$ Banach spaces, Y might not have a complement,

$$Y^* = X^* / \operatorname{Ann}(Y)$$

with the quotient norm

$$\big\| \, [y] \, \big\|_{Y^*} = \inf_{x \in \operatorname{Ann}(Y)} \|y - x\|_{Y^*}$$

Example: Every separable Banach space is a quotient of $\ell_1(\mathbb{N})$.

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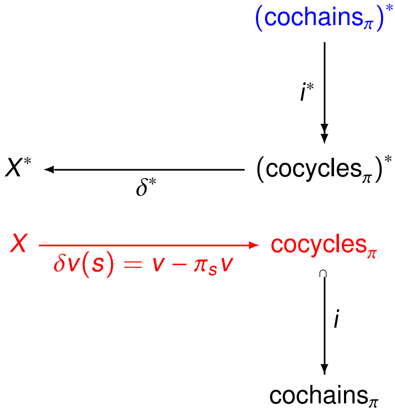

X^* is equipped with the adjoint representation,

$$\bar{\pi}_g = \pi^*_{g^{-1}}.$$

We want to show that δ is onto.

This is equivalent to δ^* having closed range.

The first step is to identify (cochains$_\pi$)*.

Theorem

If X-reflexive, π – isometric representation. Then

$$(cochains_\pi)^* \text{ is isometrically isomorphic to } cochains_{\overline{\pi}}.$$

Sketch of proof: we view cochains$_\pi$ as a complemented subspace of a larger Banach space, $\mathcal{Y}$:

$$\text{cochains}_\pi \oplus \mathcal{Z} = \mathcal{Y},$$

$$\text{cochains}_{\overline{\pi}} \oplus \overline{\mathcal{Z}} = \mathcal{Y}^*.$$

Compute to get

$$(cochains_\pi)^* = \mathcal{Y}^* / \overline{\mathcal{Z}} \ \text{ isomorphic to } \ \text{cochains}_{\overline{\pi}}$$

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We need an additional geometric condition.

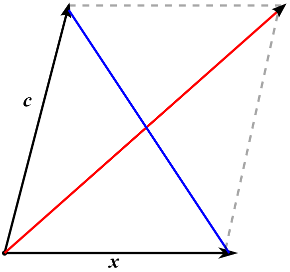
Theorem

If π is isometric then

$$\|c - x\|_{\mathcal{Y}} = \|c + x\|_{\mathcal{Y}},$$

for $c \in cochains_{\overline{\pi}}$, $x \in \overline{\mathcal{Z}}$

This is an orthogonality-type condition



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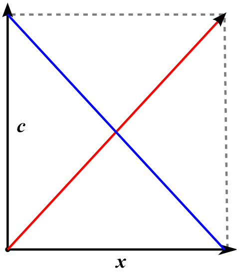

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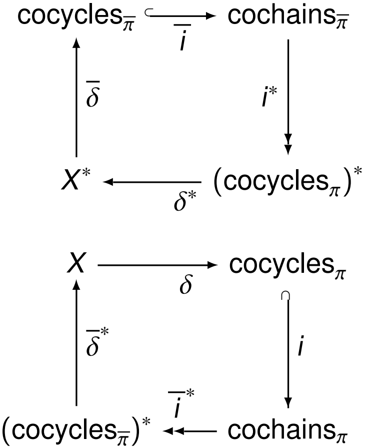


Thm 1. If $2^{1/p^*}\kappa_{p^*}(S,X) < 1$ then $\delta^*\, i^*\, \bar{i}$ has closed range.

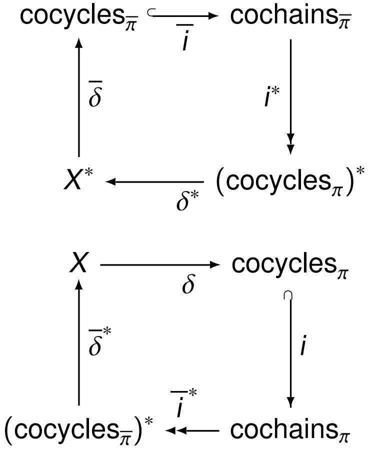
Thm 1 follows from a sequence of inequalities

It implies δ^* has closed range on image of $i^*\,\bar{i}$

The same argument for the other inequality gives: $2^{1/p}\kappa_p(S,X) < 1$ then $\overline{\delta}^*\, \overline{i}^*\, i$ has closed range

$\Rightarrow$ $\overline{i}^*\, i$ has closed range

$\Rightarrow$ $i^*\,\bar{i}$ is surjective



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Applications

We want to apply this to $X = L_p$, $p > 2$

Desired outcome: vanishing of cohomology for all L_p,

$$p \in [2, 2 + c),$$

where we can say something about c.

Remark. This cannot be improved, in the sense that we cannot expect vanishing for all $2 < p < \infty$:

1. p-Poincaré constants > 1 for p sufficiently large
2. the main theorem applies to hyperbolic groups

Difficulties: estimating p-Poincaré constants is a hard problem in analysis when $p \neq 1, 2, \infty$.

$\widetilde{A}_2$ groups

Cartwright, Młotkowski and Steger defined finitely presented groups G_q where $q = k^n$ for k - prime such that

$$\mathcal{L}(S) = \text{ incidence graph of a projective plane over a finite field}$$

In the 60ies Feit and Higman computed spectra of such incidence graphs, which implies

$$2^{-\frac{1}{2}}\kappa_2(S,\mathbb{R}) = \sqrt{\left(1 - \frac{\sqrt{q}}{q+1}\right)^{-1}} \quad \longrightarrow \quad \frac{1}{\sqrt{2}}.$$

We now want to estimate $\kappa_p(S, L_p)$ for these graphs.

Estimating the p-Poincaré constant

When $p \geq 2$, in finite dimensional spaces: $\|f\|_{\ell_p^n} \leq \|f\|_{\ell_2^n} \leq n^{1/2-1/p}\|f\|_{\ell_p^n}$.

- $\#V = 2(q^2+q+1)$,
- $\#E = 2(q^2+q+1)(q+1)$
- $\deg(s) = q+1$ for every $s \in S$

Similarly for $p^* < 2$.

Theorem

For each q=power of a prime we have

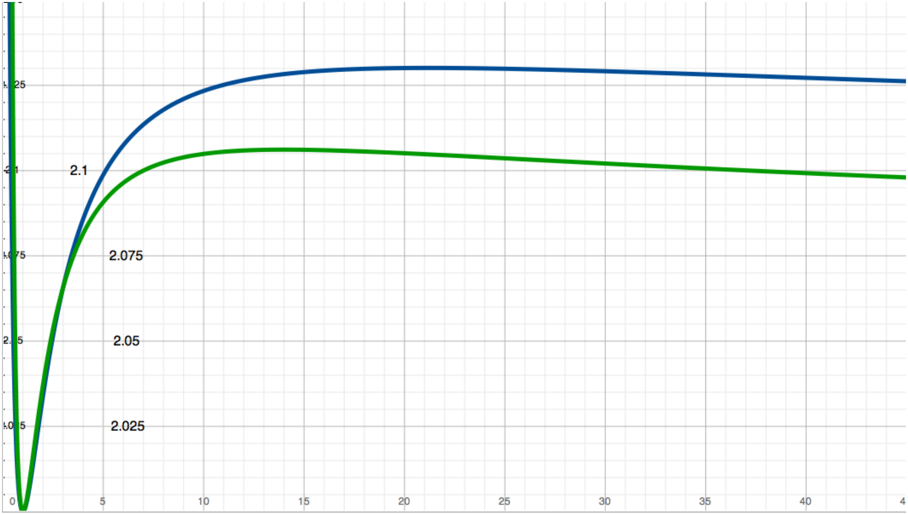
$$H^1(G_q, \pi) = 0$$

for any isometric representation π of G_q on any L_p for all

$$2 \leq p < \frac{2\ln\left(2(q^2+q+1)\right)}{\ln\left(2(q^2+q+1)\right) - \ln\sqrt{2\left(1-\frac{\sqrt{q}}{q+1}\right)}}.$$

Numerical values of p

We have $2 \leq p \leq 2.106$ and $p \to 2$ as $q \to \infty$.



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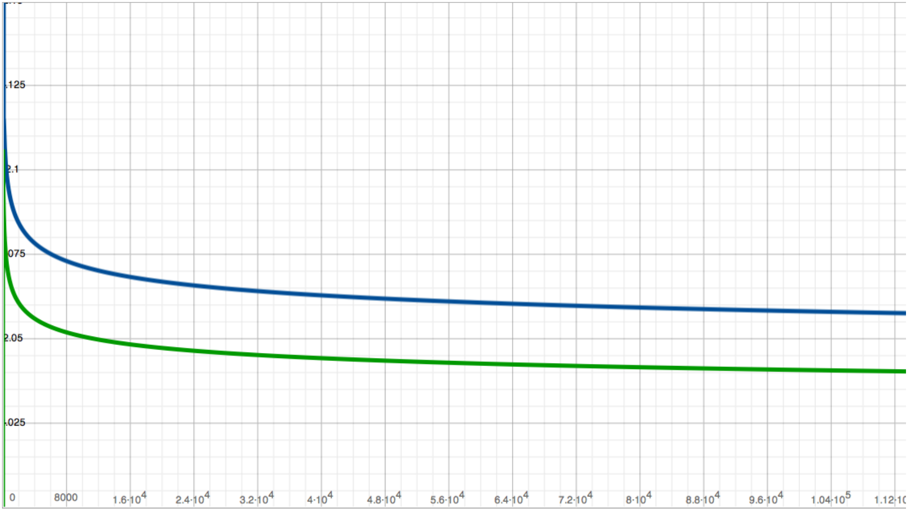

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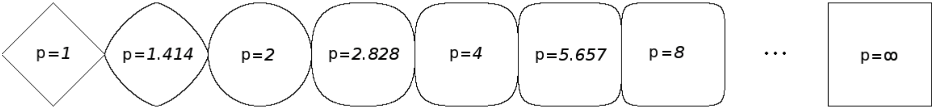

Hyperbolic groups

Żuk used the spectral conditions to prove that many hyperbolic groups have (T).

Because of randomness we cannot hope for explicit bounds on p.

Theorem (Żuk)

A group G in the density model for $1/3 < d < 1/2$ is, with probability 1, of the form

$$H \twoheadrightarrow \Gamma \quad \subseteq_{f.i.} \quad G,$$

where G is hyperbolic and H has a link graph with $2^{-1/2}\kappa_2(S,\mathbb{R}) < 1$.

Vanishing of cohomology for all isometric representations on L_p is passed on to quotients and by finite index subgroups, just as (T) is.

Corollary

With probability 1, the main theorem applies to hyperbolic groups.

Conformal dimension

Definition (Pansu)

G hyperbolic, d_V -any visual metric on ∂G.

$$\mathrm{confdim}(\partial G) = \inf \left\{ \dim_{Haus}(\partial G, d) : \, d \text{ quasi-conformally equiv. to } d_V \right\}.$$

$\mathrm{confdim}(\partial G)$ is a q.i. invariant of G, extremely hard to estimate

Bourdon-Pajot, 2003: G acts without fixed points on $\ell_p(G)$ for $p \geq \mathrm{confdim}(\partial G)$

Corollary. The main theorem gives lower bounds on $\mathrm{confdim}(\partial G)$.

Corollary

Let G be a hyperbolic group. Then for $p > \mathrm{confdim}(\partial G)$ we have

$$2^{-1/p}\kappa_p(S, X) \geq 1 \quad \textit{or} \quad 2^{-1/p^*}\kappa_{p^*}(S, X^*) \geq 1.$$

Applications to actions on the circle

Navas studied rigidity properties of diffeomorphic actions on the circle.

Vanishing of cohomology for L_p for $p > 2$ improves the differentiability class in his result.

Corollary

Let q be a power of a prime number and G_q be be the corresponding $\widetilde{A}_2$ group. Then every homomorphism $h : G \to \mathrm{Diff}_+^{1+\alpha}(S^1)$ has finite image for

$$\alpha > \frac{\frac{1}{2}\ln(2(q^2+q+1)(q+1)) - \ln(2) - \ln\left(\sqrt{1-\frac{\sqrt{q}}{q+1}}\right)}{\ln(q^2+q+1)+\ln(q+1)}.$$

Here, α is strictly less than $\frac{1}{2}$, improving for these groups the original differentiability class.

Final comments

- One more application to finite dimensional representations allows to estimate eigenvalues of the p-Laplacian on finite quotients of groups (some previous estimates using different techniques in joint work with R.I. Grigorchuk)

Q: Do $\widetilde{A}_2$ groups admit an affine isometric action on L_p, without fixed points or metrically proper, for p sufficiently large?