RAAGs in Ham

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- Negative results: L. Polterovich; Franks and Handel: A non-uniform lattice of rank ≥ 2 cannot embed in $Diff(M, \omega)$. A non-uniform (irreducible) lattice in a Lie group (different from $O(n, 1)$) cannot embed in $Diff(M, \omega)$ if $\chi(M) \leq 0$.
- Question: What happens with lattices in $O(n, 1)$?

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- $Ham(M, \omega)$ is the group of Hamiltonian symplectomorphisms. If M is a surface, $Ham(M, \omega) \subset Diff(M, \omega)$.
- Note: If M is a surface, then, as a group, $Ham(M, \omega)$ is independent of ω (Mozer); thus, $Ham(M, \omega) = Ham(M)$.

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- Examples: Free groups, free abelian groups,...
- **Theorem** (Bergeron, Haglind, Wise): If Γ is an arithmetic lattice in $O(n, 1)$ of the simplest type then a finite-index subgroup in Γ embeds in some RAAG.

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- **Corollary.** For every n there exist finite volume hyperbolic n-manifolds N (compact and not) so that $\pi_1(N)$ embeds in every Ham.
- The most difficult case is $M = S^2$.

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- **Step 3.** Smooth out the action preserving faithfulness.

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- Such ρ is probably faithful for "generic" functions H_v, but I do not know how to prove it!

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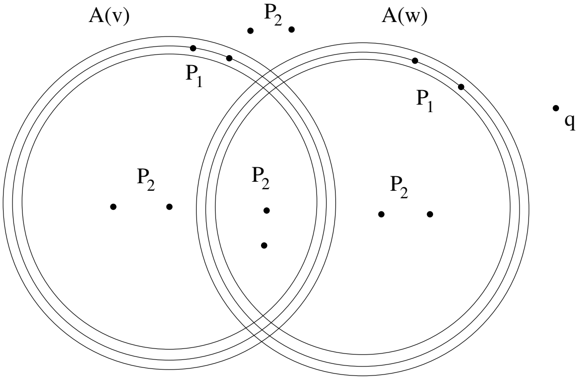

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Figure: Points p_i, q are punctures to be removed from the surface.

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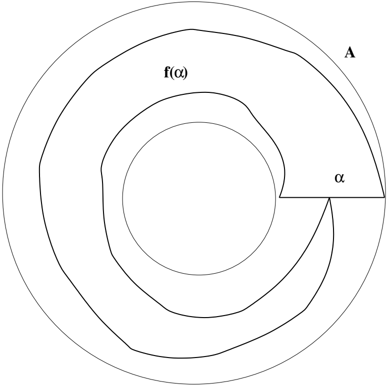

Figure: Point-pushing map (up to C^0 relative isotopy).

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- This does the job if M is S^2.
- But Γ need not be planar.
- One can show (with a bit of trickery) that if Γ admits a finite planar orbi-cover $\Lambda \to \Gamma$ then $G_\Gamma \hookrightarrow G_\Lambda$ and we are again OK.

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- Let ω_0 be the spherical area form on S^2.
- We can lift functions H_v to D and try to use ω_0 to define new time-2 Hamiltonian maps using these functions. The resulting maps preserve ω_0 on D, but ...

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- Good: The blow-up is only polynomial in z.

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- I do not know what happens for non-generic ϵ, even those close to 0.

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- Then extend the diagonal action by the identity to the rest of M.

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- For example: Take Λ which is a connected linear graph on n vertices. Is it true that for every finite graph Γ there exists an embedding $G_\Gamma \to G_\Lambda$?
- If this is the case, then, since Λ embeds in S^1, $G_\Lambda \hookrightarrow \mathit{Diff}(S^1)$, so we should also get an embedding $G_\Gamma \hookrightarrow \mathit{Diff}(S^1)$.

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- Not a single example is known.
- Note: All RAAGs are locally indicable (every f.g. subgroup has infinite abelianization). On the other hand, there are uniform lattices in $SU(2,1)$ which are not known to have virtually positive b_1.