Contracting Boundaries of CAT(0) Spaces

Ruth Charney

Dubrovnik, July 2011

Motivation

X = complete hyperbolic metric space.

Visual boundary of X:

$$\partial X = \{\text{geodesic rays } \alpha : [0, \infty) \to X\}/\sim$$

where $\alpha \sim \beta$ if they have bounded Hausdorff distance.

Topology on ∂X:

$$N(\alpha, r, \epsilon) = \{\beta \mid d(\alpha(t), \beta(t)) < \epsilon, 0 \leq t < r\}$$

Properties of ∂X, X hyperbolic

- If X is proper, then $X \cup \partial X$ is compact.
- Quasi-isometries $f : X \to Y$ induce homeomorphisms $\partial f : \partial X \to \partial Y$. In particular, ∂G is well-defined for a hyperbolic group G.
- ∂X is a visibility space, i.e. for any two points $x, y \in \partial X$, $\exists$ a geodesic γ with $\gamma(\infty) = x$ and $\gamma(-\infty) = y$.
- Nice dynamics: hyperbolic isometries $g \in Isom(X)$ act on ∂X with "north-south dynamics."

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- If X is proper, then $X \cup \partial X$ is compact.
- Quasi-isometries $f : X \to Y$ do NOT necessarily induce homeomorphisms $\partial f : \partial X \to \partial Y$, so ∂G is not well-defined for a CAT(0) group G (Croke-Kleiner).
- ∂X is a NOT a visibility space (eg. $X = \mathbb{R}^2$).
- Dynamics of $g \in Isom(X)$ acting on ∂X ???

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Certain isometries of a CAT(0) space X behave nicely. These are known as rank one isometries.

Rank one isometries

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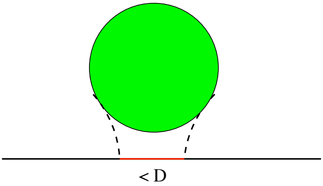
General philosophy: Rank one isometries of a CAT(0) space behave nicely because their axes behave like geodesics in a hyperbolic space.

Definition (Bestvina-Fujiwara)

A geodesic α is D-contracting if for any ball B disjoint from α, the projection of B on α has diameter at most D. A geodesic is contracting if it is D-contracting for some D.



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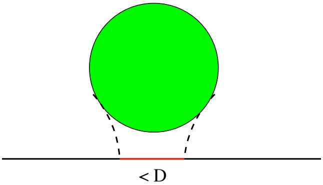
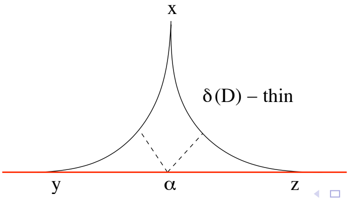


Contracting geodesics satisfy a thin triangle property.



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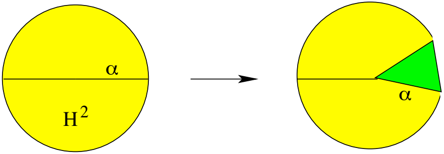

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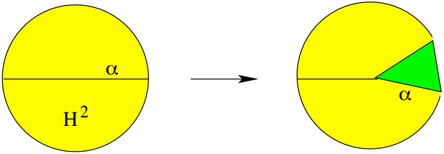

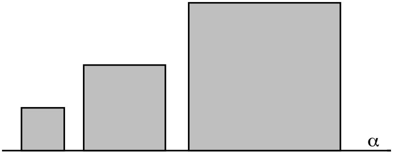

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Consider the subspace of ∂X consisting of all contracting rays.

Define the *contracting boundary* of X

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(2) $X =$ first example above, the $\partial_c X = \partial H^2 \backslash \{pt\} \cong (0, 1)$.

(3) If $X = X_1 \times X_2$, then $\partial_c X = \emptyset$

This subspace, $\partial_c X$, should behave like a hyperbolic boundary.

Properties of ∂X, for X hyperbolic:

- If X is proper, then $X \cup \partial X$ is compact.
- Quasi-isometries $f : X \to Y$ induce homeomorphisms $\partial f : \partial X \to \partial Y$. In particular, ∂G is well-defined for a hyperbolic group G.
- ∂X is a visibility space, i.e. for any two points $x, y \in \partial X$, $\exists$ a geodesic γ with $\gamma(\infty) = x$ and $\gamma(-\infty) = y$.
- hyperbolic isometries $g \in Isom(X)$ act on ∂X with "north-south dynamics."

Q: Are the analogous true for $\partial_c X$ of a CAT(0) space?

Properties of $\partial_c X$

Theorem

Suppose X is proper. The subspace of D-contracting rays is compact, hence $\partial_c X$ is σ-compact (a countable union of compact subspaces).

Proof: Follows easily from lemmas in Bestvina-Fujiwara.

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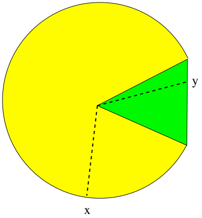

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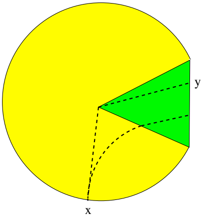

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A quasi-isometry of CAT(0) spaces $f : X \to Y$ *induces a homeomorphism* $\partial f : \partial_c X \to \partial_c Y$. *In particular,* $\partial_c G$ *is well-defined for a CAT(0) group* G.

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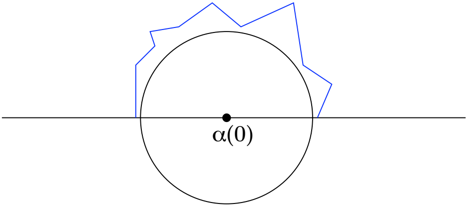
Idea of proof. Recall, a ray α is D-contracting if for any ball B disjoint from α, the projection of B on α has diameter at most D.

Problem: projection does not behave nicely under quasi-isometry. Need a characterization of contracting ray which does.

Divergence: For α a (bi-infinte) geodesic,

$$div_\alpha(r) = \inf\{\ell(p) \mid p \text{ a path in } X \backslash B(r, \alpha(0)) \text{ from } \alpha(-r) \text{ to } \alpha(r)\}$$



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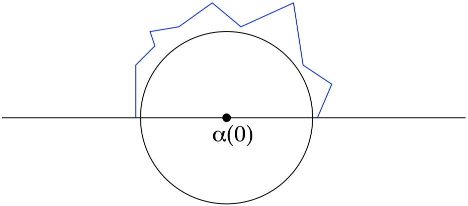

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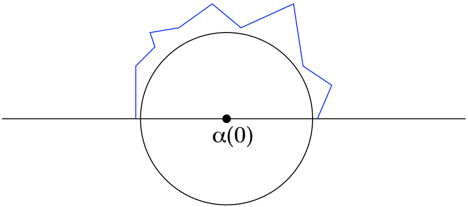

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Remark

These are different even for a bi-infinite geodesics. Eg, if $X = \mathbb{R}^2 \vee \mathbb{R}^2$ and α passes through 0, then $div_\alpha(r) = \infty$, while $\underline{div}_\alpha(r) = \pi r$.

Lemma

For a ray α in X, TFAE

1. $\underline{div}_\alpha$ *is super-linear.*
2. $\underline{div}_\alpha$ *is at least quadratic.*
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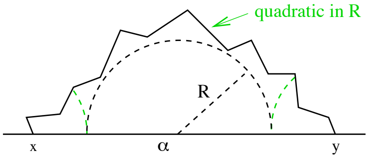
Step 1: Show $f(\alpha)$ stays bounded distance from some geodesic ray β in Y.



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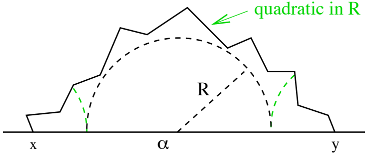

Step 2: Show $\underline{div}_\beta(r) \asymp \underline{div}_\alpha(r)$, hence α contracting $\Rightarrow$ β contracting.

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Caprace-Sageev: in very general CAT(0) cube complexes:

X has a rank one isometry $\Leftrightarrow$
X has a pair of strongly separated walls.

Want to characterize contracting rays in terms of strongly separated walls.

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Too strong: suppose α lies in a wall H. Then no two walls crossed by α are strongly separated.

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Theorem

X as above. Then a geodesic ray α in X is contracting $\Leftrightarrow \exists C > 0, k \in \mathbb{N}$, such that any segment of α of length C crosses a pair of k-separated walls.

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Question

For CAT(0) cube complexes, what is the relation between $\partial_c X$ and the Poisson boundary described in Sageev's talk?