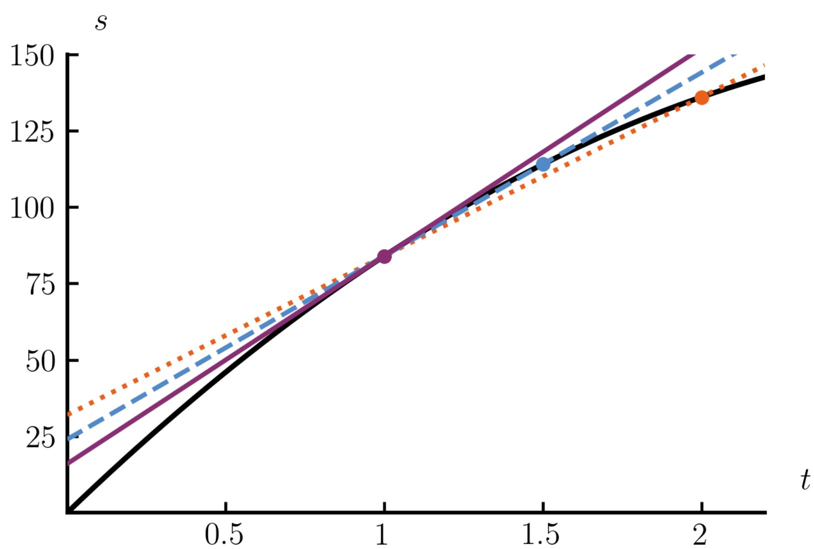
Why We Need Limits

In this lesson, we will explore two problems which require limits to solve: instantaneous velocity and tangent lines.

Suppose you travel 130 miles down the highway in two hours. Your **average velocity** over this time interval is $\frac{130}{2} = 65$ mph. But the **instantaneous velocity** indicated by the speedometer changes each moment and is not always 65 mph.

Example 1: A ball is launched upward from the ground with a velocity of 100 ft/s. Its height in feet after t seconds is $s(t) = 100t - 16t^2$. Find its instantaneous velocity after 1 second.



Solution:

The average velocity on $[1, t]$ is $v_{\text{ave}} = \frac{s(t) - s(1)}{t - 1}$.

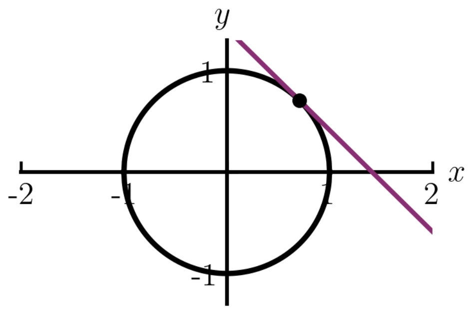
When $t = 2$, $v_{\text{ave}} = \frac{136 - 84}{2 - 1} = 52$.

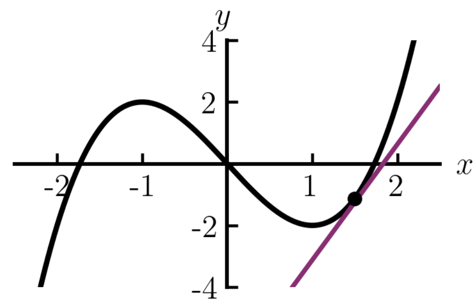
When $t = 1.5$, $v_{\text{ave}} = \frac{114 - 84}{1.5 - 1} = 60$.

$v_{\text{inst}} = \lim_{t \to 1} v_{\text{ave}} = \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} = 68$.

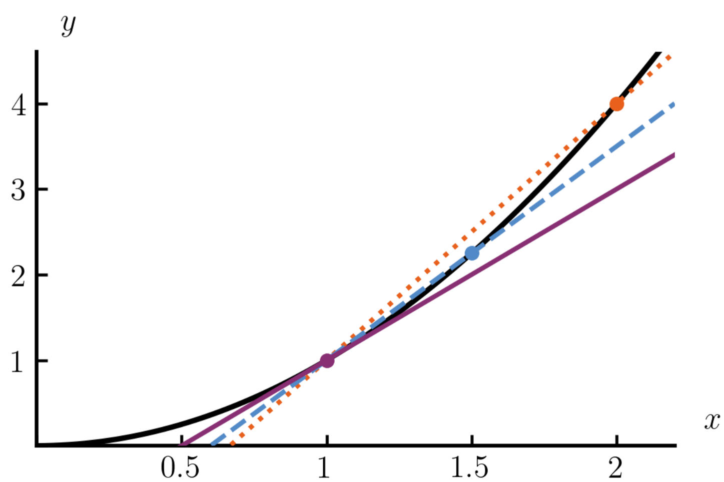
t	v_{ave}
2	52
1.5	60
1.1	66.4
1.01	67.84
1.001	67.984

For a circle, a tangent line is a line which touches the circle at exactly one point. The tangent line has the same slope as the circle at that point. We might wonder how we can find lines like these for other curves. A **tangent line** has the same direction as the curve at the specified point.





Example 2: Find the slope of the tangent line to $y = x^2$ at the point $P = (1, 1)$.



Solution:

The slope of the secant line is $m_{\text{sec}} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Let $x_1 = 1$. When $x_2 = 2$, $m_{\text{sec}} = \dfrac{4-1}{2-1} = 3$.

When $x_2 = 1.5$, $m_{\text{sec}} = \dfrac{2.25 - 1}{1.5 - 1} = 2.5$.

$m_{\tan} = \lim\limits_{x_2 \to 1} m_{\text{sec}} = \lim\limits_{x_2 \to 1} \dfrac{x_2^2 - 1^2}{x_2 - 1} = 2$.

x_2	m_{sec}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001