The Shell Method

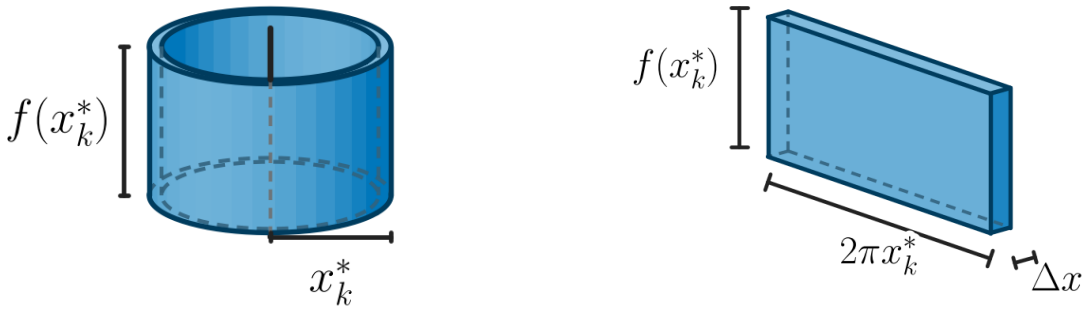
In this lesson, we will find the volume of a solid of revolution with the shell method, and practice using shells about horizontal and vertical lines.

For some solids, disks and washers can be difficult or impossible to use. Using cylindrical shells may be more efficient. Revolving vertical strips about the y-axis creates cylindrical shells.



Cut the interval $[a, b]$ into n equal pieces with width $\Delta x = \dfrac{b-a}{n}$, and sample points x_k^*.

Unrolling a thin shell turns it into a rectangular sheet. The width of the sheet is the circumference $2\pi x_k^*$, the height is $f(x_k^*)$, and the thickness is Δx.



$$V \approx \sum_{k=1}^{n} \underbrace{2\pi x_k^*}_{\text{circumference}} \underbrace{f(x_k^*)}_{\text{height}} \underbrace{\Delta x}_{\text{thickness}}$$

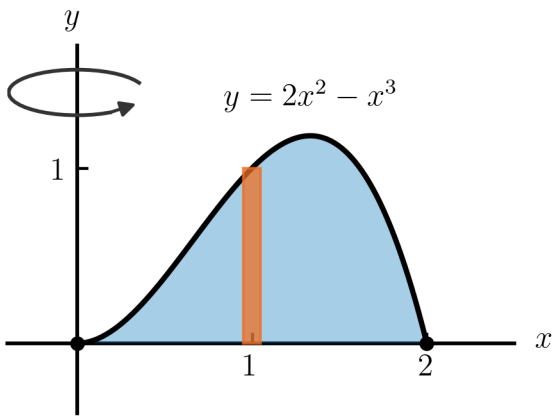
To get the exact volume, we take the limit as $n \to \infty$.

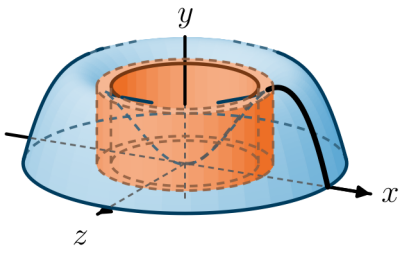
$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x$$

$$V = \int_a^b 2\pi x f(x)\, dx$$

This is the **Shell Method**. In this formula, the radius is x, the distance to the y-axis. If the solid revolves around a different line, we can adjust the circumference, height, and thickness to get the correct formula.

Example 1: Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$, for $0 \leq x \leq 2$, about the y-axis.





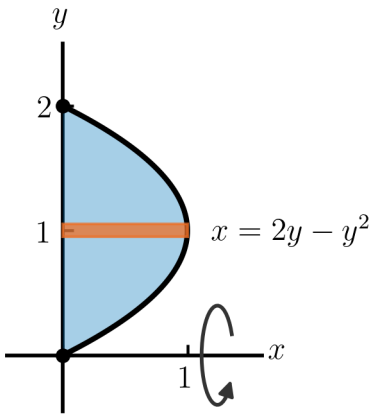
Solution:

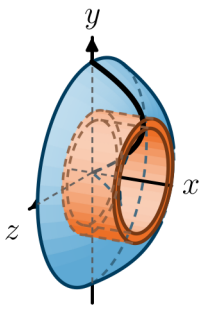
Using washers would require solving the equation for x. Instead, we will use shells. The shells are vertical, since the axis is the y-axis. The strip at x has circumference $2\pi x$, height $2x^2 - x^3$, and thickness dx.

For this region, x runs from 0 to 2, so the volume is

$$\begin{aligned} V &= \int_0^2 2\pi x(2x^2 - x^3)\,dx \\ &= 2\pi \int_0^2 (2x^3 - x^4)\,dx \\ &= 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 \\ &= 2\pi \left(\frac{1}{2} \cdot 16 - \frac{1}{5} \cdot 32\right) \\ &= \boxed{\frac{16\pi}{5}} \end{aligned}$$

Example 2: Find the volume of the solid obtained by rotating the region bounded by $x = 2y - y^2$ and $x = 0$, for $0 \le y \le 2$, about the x-axis.





Solution:

Using washers would require solving $x = 2y - y^2$ for y in terms of x and getting the two solutions with square roots. Instead, we will use shells. The shells are horizontal, since we are revolving around the x-axis.

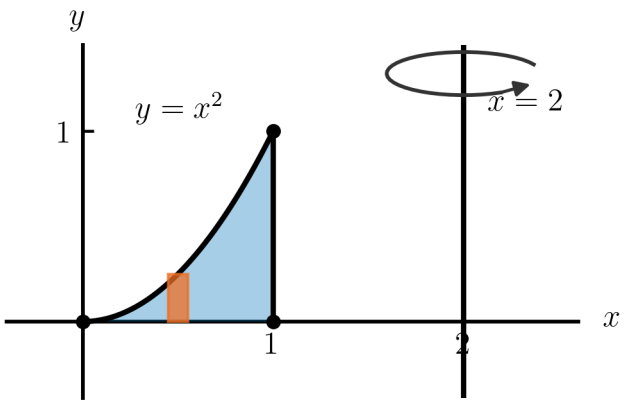
The radius is the distance to the x-axis, and the height is the horizontal length of the strip. At height y, the circumference is $2\pi y$, the height is $2y - y^2$, and the thickness is dy.

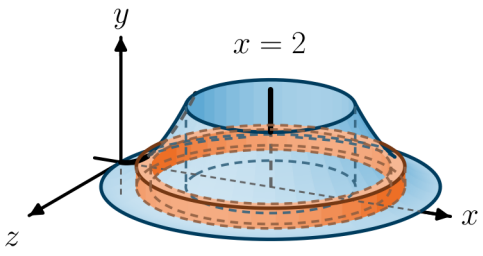
For this region, y runs from 0 to 2, so the volume is

$$\begin{aligned} V &= \int_0^2 2\pi y(2y - y^2)\, dy \\ &= 2\pi \int_0^2 (2y^2 - y^3)\, dy \\ &= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 \\ &= 2\pi \left(\frac{2}{3} \cdot 8 - \frac{1}{4} \cdot 16\right) \\ &= \boxed{\frac{8\pi}{3}} \end{aligned}$$

In Example 1, the axis was the y-axis, and in Example 2, the axis was the x-axis. Next, try this interactive problem on the website to use the shell method about the line $x = 2$.

Interactive Problem 1: Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 0$, and $x = 1$ about the line $x = 2$.





(a) For a fixed x in $[0, 1]$, the radius of the cylindrical shell is $2 - x$.

(b) For a fixed x in $[0, 1]$, the height of the cylindrical shell is x^2.

(c) The volume of the solid is $V = \int_0^1 2\pi(2 - x)x^2\,dx$.

(d) The volume of the solid is

$$\int_0^1 2\pi(2 - x)x^2\,dx = 2\pi\int_0^1 (2x^2 - x^3)\,dx = 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 2\pi\left(\frac{2}{3} - \frac{1}{4}\right) = \boxed{\frac{5\pi}{6}}.$$