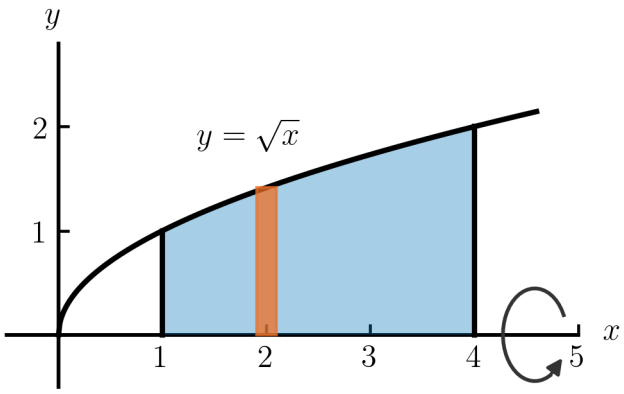
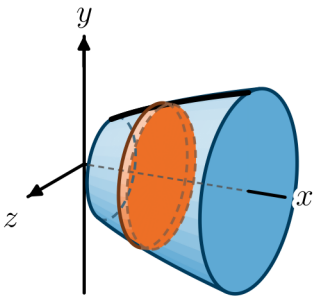
The Disk Method

In this lesson, we will find the volume of a solid of revolution with the disk method, and practice using disks about horizontal and vertical lines.

A **solid of revolution** is the solid formed by rotating a region about a line.



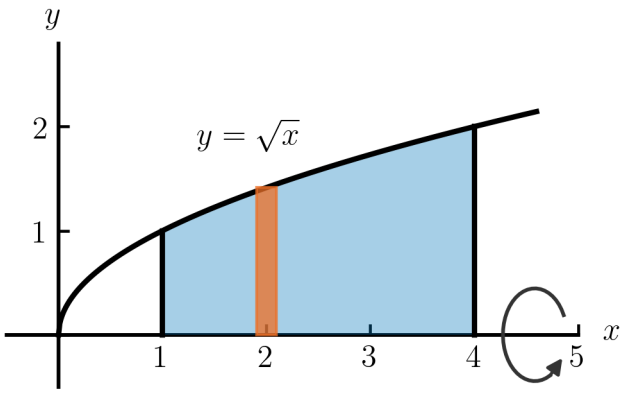


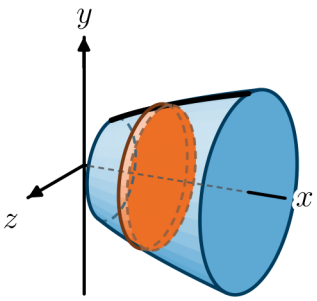
When the region lies along the line, the cross sections perpendicular to the line are **circular disks**. The volume is the integral of the cross-sectional area. In this case, the cross sections are circles with radius $r(x)$ so the area is $A(x) = \pi\big[r(x)\big]^2$. We get:

$$V = \int_a^b \pi\big[r(x)\big]^2\,dx.$$

This is the **disk method**. The integral can be written in terms of y to rotate about vertical lines.

Example 1: Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$, and $x = 4$ about the x-axis.





Solution:

The disks are perpendicular to the x-axis. At x, the radius goes from $y = 0$ to $y = \sqrt{x}$, so the radius is

$$r(x) = \sqrt{x} - 0 = \sqrt{x}.$$

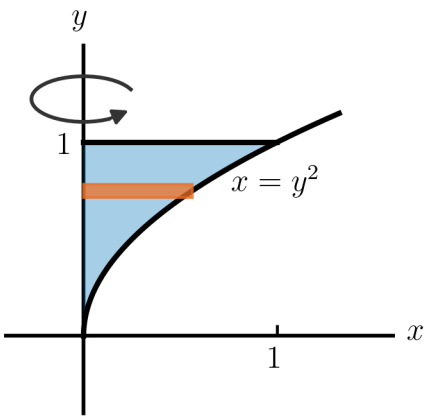
The cross-sectional area is:

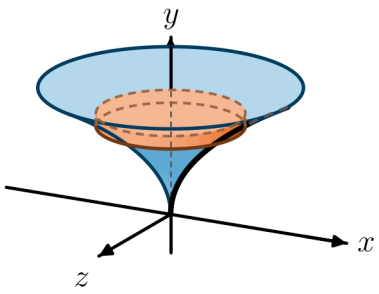
$$A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x.$$

For this region, x runs from 1 to 4. The volume is:

$$\begin{aligned} V &= \int_1^4 \pi x \, dx \\ &= \left[\frac{\pi x^2}{2}\right]_1^4 \\ &= \frac{\pi \cdot 16}{2} - \frac{\pi \cdot 1}{2} \\ &= 8\pi - \frac{\pi}{2} \\ &= \boxed{\frac{15\pi}{2}} \end{aligned}$$

Example 2: Find the volume of the solid obtained by rotating the region $0 \leq y \leq 1, 0 \leq x \leq y^2$ about the y-axis.





Solution:

The disks are perpendicular to the y-axis. At y, the radius goes from $x = 0$ to $x = y^2$, so the radius is:

$$r(y) = y^2 - 0 = y^2$$

The cross-sectional area is:

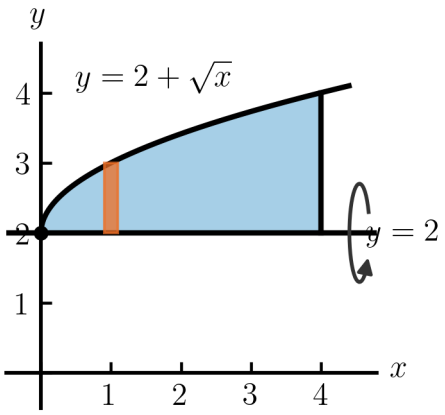
$$A(y) = \pi\left(y^2\right)^2 = \pi y^4$$

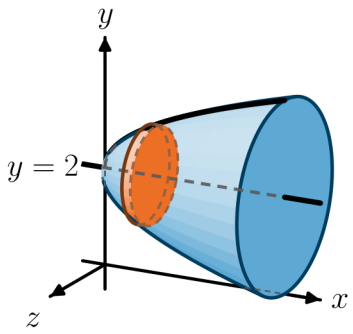
For this region, y runs from 0 to 1. The volume is:

$$\begin{aligned} V &= \int_0^1 \pi y^4 \, dy \\ &= \left[\frac{\pi y^5}{5}\right]_0^1 \\ &= \frac{\pi}{5} - 0 \\ &= \boxed{\frac{\pi}{5}} \end{aligned}$$

In Example 1, the axis was the x-axis, and in Example 2, the axis was the y-axis. Next, try this interactive problem on the website to use the disk method about the line $y = 2$.

Interactive Problem 1: Find the volume of the solid obtained by rotating the region bounded by $y = 2 + \sqrt{x}$, $y = 2$, and $x = 4$ about the line $y = 2$.





(a) The volume of the solid in terms of the cross-sectional area is $V = \int_0^4 A(x)\,dx$.

(b) For a fixed x in $[0, 4]$, the radius of the disk is $r(x) = \left(2 + \sqrt{x}\right) - 2 = \sqrt{x}$.

(c) The cross-sectional area at a fixed x in $[0, 4]$ is $A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x$.

(d) The volume of the solid is $\int_0^4 \pi x\,dx = \left[\frac{\pi x^2}{2}\right]_0^4 = \frac{\pi \cdot 16}{2} - 0 = 8\pi$.