The Derivative as a Function

We will define the derivative as a function and practice using the limit definition.

We have seen how to find the derivative of a function at a point. At each point a where the derivative exists, we get the value $f'(a)$. Now we will view f' as a function. $f'(x)$ is the derivative of f at x, which is the instantaneous rate of change (slope of the tangent line) of f at x.

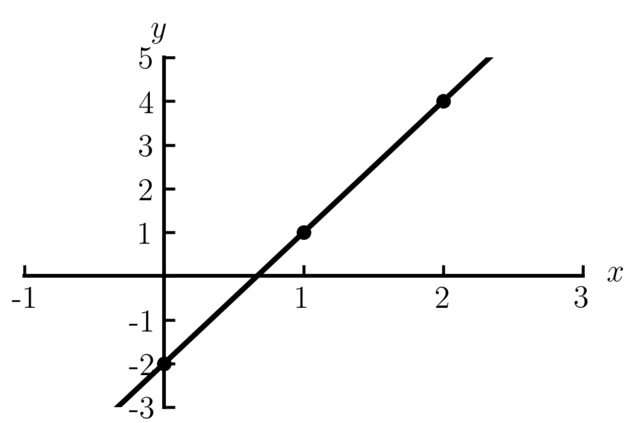
Definition The Derivative as a Function: The derivative of f is the function f' defined by $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, if the limit exists. If $f'(x)$ exists, we say that f is differentiable at x. If f is differentiable at every point in an open interval I, we say that f is differentiable on I.

Example 1: Let $f(x) = 3x - 2$. Find $f'(x)$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h)-2-(3x-2)}{h} \\ &= \lim_{h\to 0} \frac{3x+3h-2-3x+2}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 = 3 \end{aligned}$$

The derivative of $f(x) = 3x - 2$ is the constant function $f'(x) = 3$. Since $f(x)$ is a line with slope 3, the slope of the tangent line is always 3 (the same line).



There is another common notation for the derivative of f. If we write $\Delta x = x_2 - x_1$ for the change in x, and $\Delta y = y_2 - y_1$ for the change in y, then the slope of the secant line is $\dfrac{\Delta y}{\Delta x}$. The derivative is $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ which we write as $\dfrac{dy}{dx}$. This is called the Leibniz notation for the derivative.

Here are some of the ways to write the derivative of $y = f(x)$:

$$f'(x), \quad y', \quad \frac{df}{dx}, \quad \frac{dy}{dx}, \quad \frac{d}{dx}(f(x)), \quad \frac{d}{dx}(y), \quad Df(x), \quad D_x f(x)$$

These all mean the same thing: the derivative of $f(x)$.

Example 2: Let $y = f(x) = \sqrt{x}$. Find $\dfrac{dy}{dx}$.

Solution: To find the derivative, we use the limit definition and rationalize the numerator by multiplying by the conjugate.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

The derivative does not exist at $x = 0$. For $x > 0$, the derivative is $\dfrac{1}{2\sqrt{x}}$.

Example 3: Let $f(x) = \dfrac{x}{x+1}$. Find $f'(x)$.

Solution: In the limit definition, we get nested fractions. Instead of dividing by h, we will multiply by $\dfrac{1}{h}$ to avoid nested fractions.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h} \\
&= \lim_{h\to 0} \left(\frac{x+h}{x+h+1} - \frac{x}{x+1} \right) \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{x^2 + x + hx + h - x^2 - xh - x}{(x+h+1)(x+1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{h}{(x+h+1)(x+1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} \\
&= \frac{1}{(x+0+1)(x+1)} = \frac{1}{(x+1)^2}
\end{aligned}$$