Sigma Notation

In this lesson, we will explore sigma notation, and use summation formulas to evaluate sums.

When approximating the area under a curve with rectangles, we end up with long sums of terms. **Sigma notation** is a way to write sums more compactly. For example,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = \sum_{i=1}^{7} i$$

The symbol $\sum$ is the Greek letter sigma, which stands for sum. The index i begins at $i = 1$, increases by 1 each time, and ends at $i = 7$. The variable i is the index of summation, which is a placeholder and not part of the evaluated sum.

Example 1: Write $\sum\limits_{k=1}^{5}(2k+1)$ in expanded form.

Solution: Substitute $k = 1, 2, 3, 4, 5$ into $2k + 1$:

$$\begin{aligned}\sum_{k=1}^{5}(2k+1) &= (2\cdot 1+1)+(2\cdot 2+1)+(2\cdot 3+1)+(2\cdot 4+1)+(2\cdot 5+1)\\ &= 3+5+7+9+11 = 35\end{aligned}$$

Example 2: Write the sum $0 + 1 + 4 + 9 + 16 + \ldots + 100$ in sigma notation.

Solution: Each term is a perfect square: $0 = 0^2$, $1 = 1^2$, $4 = 2^2$, and so on, up to $100 = 10^2$. The index runs from 0 to 10:

$$0 + 1 + 4 + 9 + 16 + \cdots + 100 = \sum_{k=0}^{10} k^2$$

Example 3: Write $\sum\limits_{i=1}^{n}(3i-2)$ in expanded form.

Solution: Substitute $i = 1, 2, 3, \ldots, n$ into $3i - 2$:

$$\begin{aligned}\sum_{i=1}^{n}(3i-2) &= (3\cdot 1-2)+(3\cdot 2-2)+(3\cdot 3-2)+\cdots+(3n-2)\\ &= 1+4+7+\cdots+(3n-2)\end{aligned}$$

We can factor out constants from a sum, or split a sum into two sums:

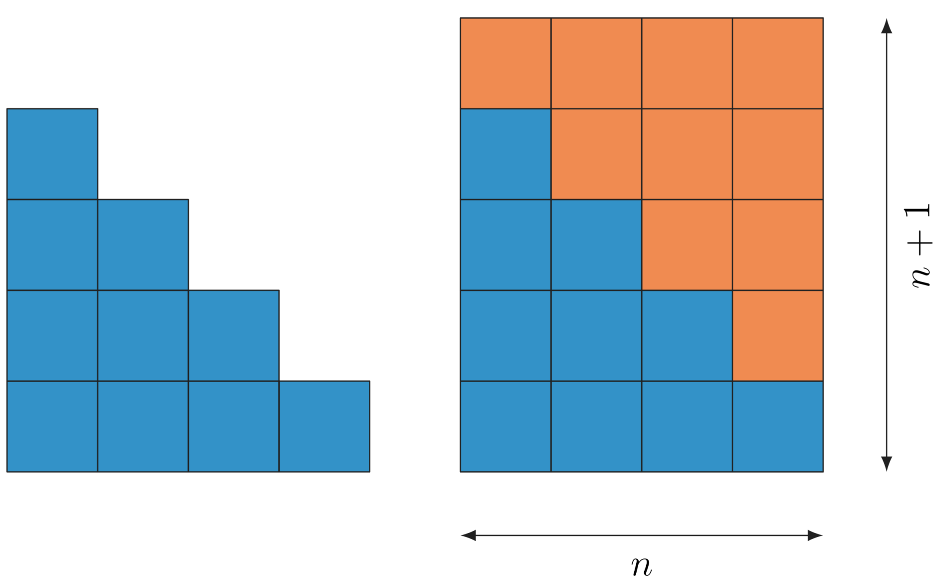
Theorem Properties of Summation: Let c be a constant and n a positive integer. Then:

$$\sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$$
$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$
$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Summation formulas help us evaluate sums. For example, if we add the same constant c from $i = 1$ to n,

$$\sum_{i=1}^{n} c = \underbrace{c + c + \ldots + c}_{n \text{ times}} = nc$$

We can find $\sum_{i=1}^{n} i$ by drawing a staircase diagram:



The blue staircase has $\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$ squares. Two copies of this staircase form an $n \times (n+1)$ rectangle, so $2 \sum_{i=1}^{n} i = n(n+1)$. Then $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Here are some common summation formulas:

Theorem Summation Formulas: Let n be a positive integer. Then:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 4: Evaluate $\displaystyle\sum_{i=1}^{n}(i^2 + 2i - 3)$.

Solution: Split the sum and use the summation formulas:

$$\begin{aligned}
\sum_{i=1}^{n}(i^2 + 2i - 3) &= \sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 3 \\
&= \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} - 3n \\
&= \frac{n(n+1)(2n+1)}{6} + n(n+1) - 3n
\end{aligned}$$