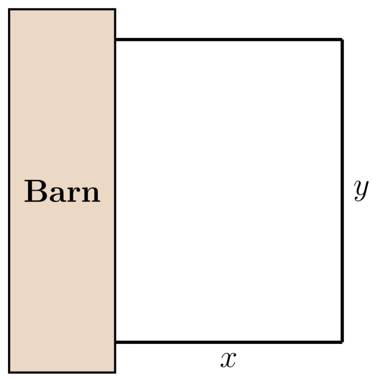
Optimization

In this lesson, we will explore optimization problems and solve two examples relating to area and perimeter.

Optimization problems ask us to maximize or minimize a quantity. For example, we might want to maximize area or minimize cost. We call this quantity the **objective function**. Conditions that the variables must satisfy are called **constraints**.

Example 1: A farmer has 1200 feet of fencing and wants to enclose a rectangular field, with one side of the rectangle up against the barn. No fence is needed for that side. What are the dimensions of the field with maximum area?



1) Label variables, identify constraints and objective:

Let x be each side perpendicular to the barn, and let y be the side parallel to the barn.

Constraint (total fencing is 1200 feet): $2x + y = 1200$.

Objective: maximize area $A = xy$.

2) Write objective as a function of one variable:

From the constraint $2x + y = 1200$, we can solve for y: $y = 1200 - 2x$.

Substituting into the objective function $A = xy$, we get: $A(x) = x(1200 - 2x) = 1200x - 2x^2$

3) Optimize the objective function: We know x is between 0 and 600, since the total fencing is 1200 feet. We can use the Closed Interval Method to find the absolute maximum by checking the endpoints and any critical points.

$$\begin{aligned} A'(x) &= 1200 - 4x = 0 \\ x &= 300 \end{aligned}$$

When $x = 300$, $y = 1200 - 2(300) = 600$, and the area is $A = 300 \cdot 600 = 180{,}000$ square feet.

We also check the endpoints $x = 0$ and $x = 600$. In both cases, the area is 0.

Therefore, the dimensions that maximize area are $x = 300$ ft by $y = 600$ ft.

Example 2: Of all rectangles with area 100, which one has minimum perimeter?

1) Label variables, identify constraints and objective:

Let the side lengths of the rectangle be x and y.

Constraint (area): $xy = 100$.

Objective: minimize perimeter $P = 2x + 2y$.

2) Write objective as a function of one variable:

From the constraint $xy = 100$, we can solve for y: $y = \dfrac{100}{x}$.

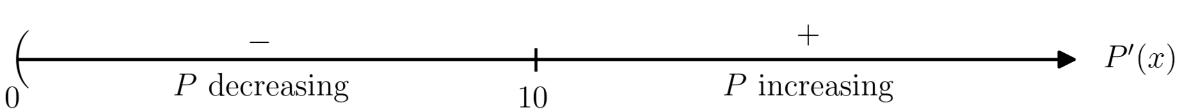
Substituting into the objective function $P = 2x + 2y$, we get: $P(x) = 2x + 2 \cdot \dfrac{100}{x} = 2x + \dfrac{200}{x}$

3) Optimize the objective function: We know x must be at least 0, but there is no upper bound so we cannot use the Closed Interval Method. To find the absolute minimum, we can construct a sign line for P'.

$$\begin{aligned} P'(x) &= 2 - \frac{200}{x^2} = 0 \\ \frac{200}{x^2} &= 2 \\ x^2 &= 100 \\ x &= 10 \end{aligned}$$

When $x = 10$, $y = \dfrac{100}{10} = 10$, and the perimeter is $P = 2(10) + 2(10) = 40$.

To check that this is the absolute minimum, we can construct a sign line for P'.



The perimeter decreases until $x = 10$, and then increases. So, the rectangle with minimum perimeter is the $x = 10$ by $y = 10$ square.