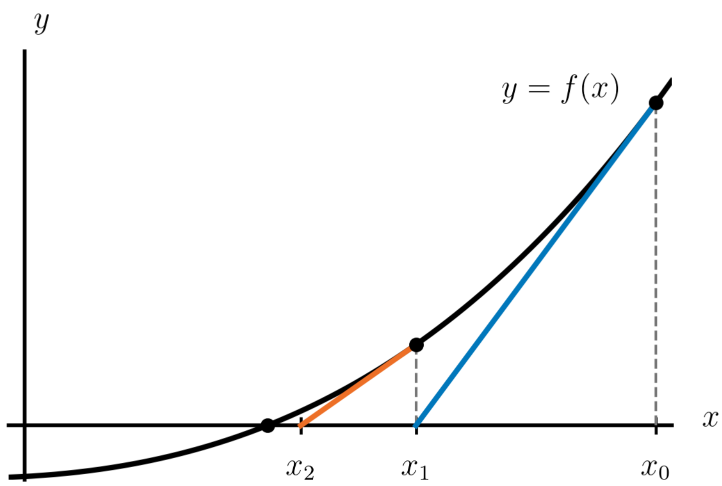
Newton’s Method

In this lesson, we will approximate roots of functions using Newton’s Method and see how it can go wrong in some cases.

A common problem in math is finding the roots, or zeros, of a function: the values x that make $f(x) = 0$. Sometimes we can find the roots exactly, but often we must approximate them instead.

The idea of Newton’s Method is to start with an estimate x_0. We find the tangent line to f at x_0, and see where the tangent lines intersects the x-axis. This is often a better estimate than x_0.



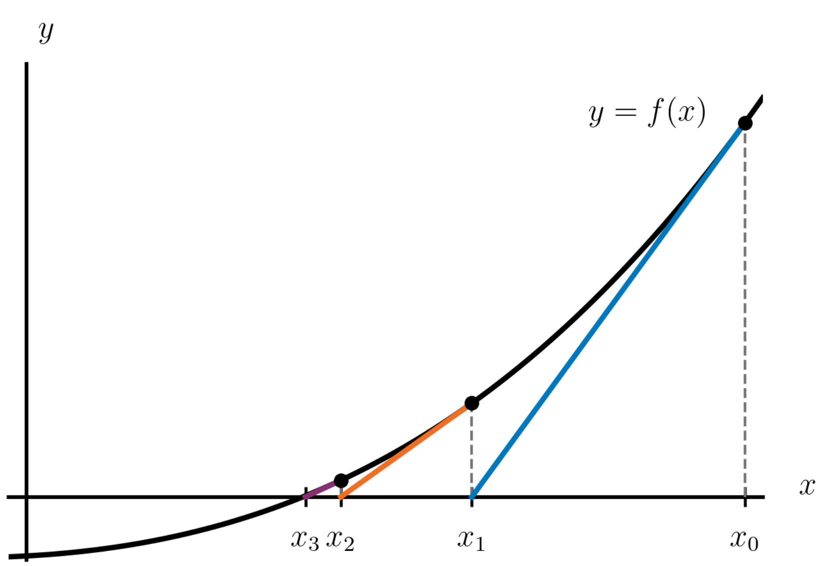
The tangent line to f at $(x_0, f(x_0))$ is given by To find the x-intercept, we set $y = 0$ and solve for x:

This gives the new estimate x_1. Then we can repeat the process, using x_1 instead of x_0.

Newton’s Method: Let f be a differentiable function.

1. Start with an estimate x_0 close to the root.
2. For each $n = 0, 1, 2, \ldots$, let $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.
3. Stop once x_{n+1} and x_n share a certain number of decimal places.

Example 1: Use Newton's Method to approximate the root of $f(x) = x^3 + 3x^2 + x - 3$ accurate to 5 decimal places, starting from $x_0 = 2$.



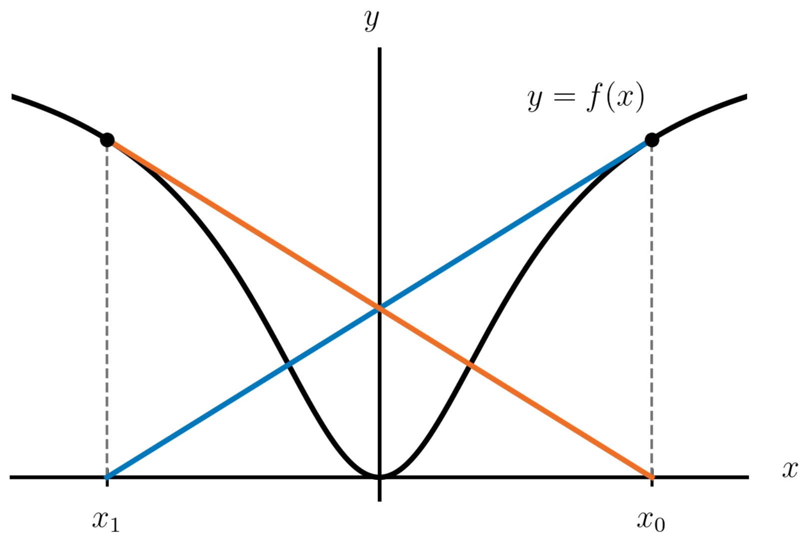
Example 2: Use Newton's Method to approximate $\sqrt{5}$ accurate to 4 decimal places.

Example 3: Use Newton's Method to approximate the positive solution to $\sin x = \frac{x}{2}$, accurate to 4 decimal places.

When Newton's Method Goes Wrong: For $f(x) = \dfrac{3x^2}{1+3x^2}$, we can solve for the only real root $x = 0$ by setting the numerator equal to 0. But, this function provides some examples of how Newton's Method can fail to converge to the root.

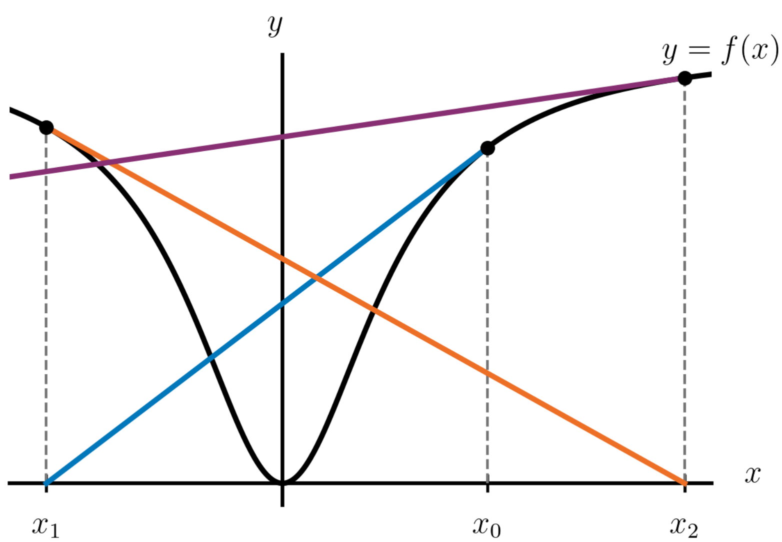
Oscillating Approximations: Starting from $x_0 = 1$, the approximations alternate between 1 and -1 forever, never converging to the root at $x = 0$:

n	x_n
0	
1	
2	
3	
4	
5	
$\vdots$	$\vdots$



Diverging Approximations: Starting from $x_0 = 1.05$, the approximations diverge, moving farther and farther from the root at $x = 0$:

n	x_n
0	
1	
2	
3	
4	



These examples show two ways Newton's Method may fail. The approximations may oscillate or diverge, never converging to the root.