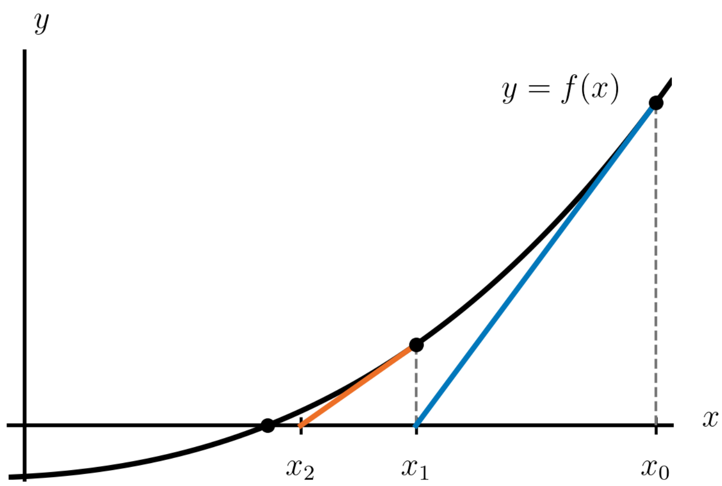
Newton's Method

In this lesson, we will approximate roots of functions using Newton's Method and see how it can go wrong in some cases.

A common problem in math is finding the roots, or zeros, of a function: the values x that make $f(x) = 0$. Sometimes we can find the roots exactly, but often we must approximate them instead.

The idea of Newton's Method is to start with an estimate x_0. We find the tangent line to f at x_0, and see where the tangent lines intersects the x-axis. This is often a better estimate than x_0.



The tangent line to f at $(x_0, f(x_0))$ is given by $y - f(x_0) = f'(x_0)(x - x_0)$. To find the x-intercept, we set $y = 0$ and solve for x:

$$\begin{aligned} 0 - f(x_0) &= f'(x_0)(x - x_0) \\ -\frac{f(x_0)}{f'(x_0)} &= x - x_0 \\ x &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

This gives the new estimate x_1. Then we can repeat the process, using x_1 instead of x_0.

Newton's Method: Let f be a differentiable function.

1. Start with an estimate x_0 close to the root.
2. For each $n = 0, 1, 2, \ldots$, let $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.
3. Stop once x_{n+1} and x_n share a certain number of decimal places.

Example 1: Use Newton's Method to approximate the root of $f(x) = x^3 + 3x^2 + x - 3$ accurate to 5 decimal places, starting from $x_0 = 2$.

Solution: The derivative is $f'(x) = 3x^2 + 6x + 1$. Starting from $x_0 = 2$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2^3 + 3(2)^2 + 2 - 3}{3(2)^2 + 6(2) + 1} = 2 - \frac{19}{25} = 1.24.$$

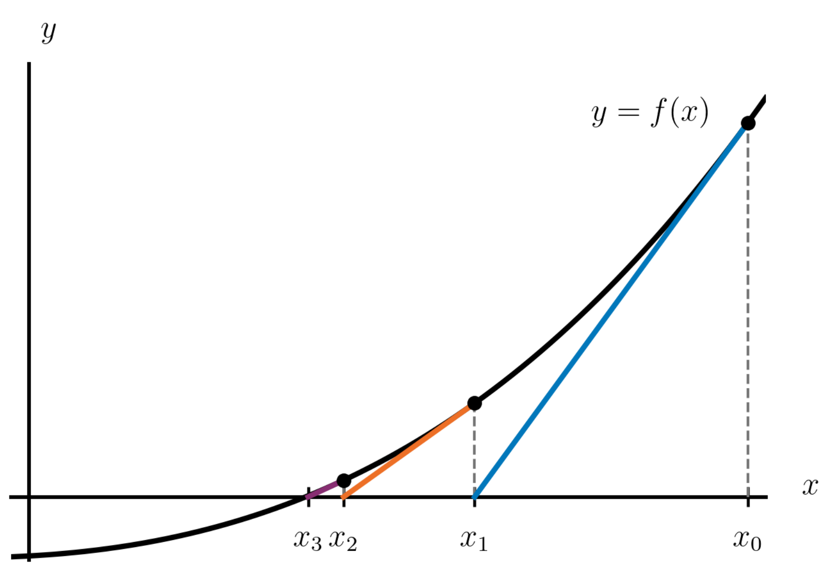
Now we use $x_1 = 1.24$ in the next iteration to get x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.24 - \frac{f(1.24)}{f'(1.24)} = 1.24 - \frac{1.24^3 + 3(1.24)^2 + 1.24 - 3}{3(1.24)^2 + 6(1.24) + 1} \approx 0.87537.$$

Repeating this process, we get:

$$\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.77656 \\
x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.76933 \\
x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} \approx 0.76929 \\
x_6 &= x_5 - \frac{f(x_5)}{f'(x_5)} \approx 0.76929
\end{aligned}$$

Since x_5 and x_6 agree to 5 decimal places, we will stop and estimate the root as $x \approx 0.76929$.



Example 2: Use Newton's Method to approximate $\sqrt{5}$ accurate to 4 decimal places.

Solution: Since $\sqrt{5}$ is a root of $f(x) = x^2 - 5$, we approximate $\sqrt{5}$ by finding a root of f. The derivative is $f'(x) = 2x$. Starting from $x_0 = 2$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^2 - 5}{2(2)} = 2 + \frac{1}{4} = 2.25.$$

Now we use $x_1 = 2.25$ in the next iteration to get x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.25 - \frac{2.25^2 - 5}{2(2.25)} = 2.25 - \frac{1}{72} \approx 2.23611.$$

Repeating this process, we get:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.23607$$

Since x_2 and x_3 agree to 4 decimal places, we stop and estimate $\sqrt{5} \approx 2.2361$.

Example 3: Use Newton's Method to approximate the positive solution to $\sin x = \frac{x}{2}$, accurate to 4 decimal places.

Solution: The positive solutions to $\sin x = \frac{x}{2}$ are the positive roots of $f(x) = \sin x - \frac{x}{2}$. We have $f'(x) = \cos x - \frac{1}{2}$. Starting from $x_0 = \frac{\pi}{2}$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{\pi}{2} - \frac{1 - \frac{\pi}{4}}{-\frac{1}{2}} = 2.$$

Now we use $x_1 = 2$ to get x_2, and so on:

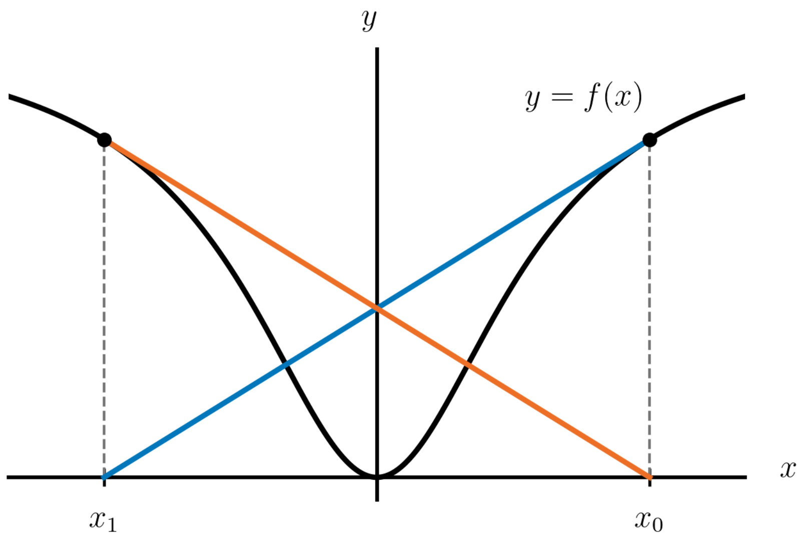
$$\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{\sin(2) - 1}{\cos(2) - \frac{1}{2}} \approx 1.90100 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.89551 \\
x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \approx 1.89549
\end{aligned}$$

Since x_3 and x_4 agree to 4 decimal places, we stop and estimate the positive solution as $x \approx 1.8955$.

When Newton's Method Goes Wrong: For $f(x) = \dfrac{3x^2}{1+3x^2}$, we can solve for the only real root $x = 0$ by setting the numerator equal to 0. But, this function provides some examples of how Newton's Method can fail to converge to the root.

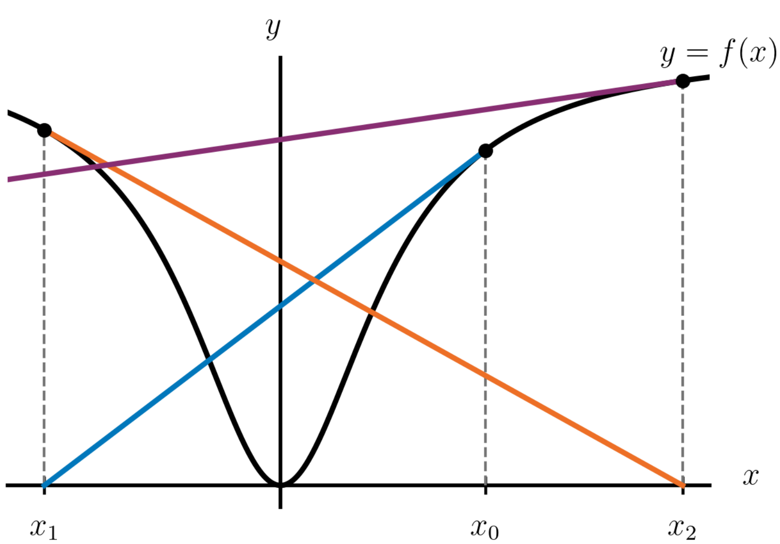
Oscillating Approximations: Starting from $x_0 = 1$, the approximations alternate between 1 and -1 forever, never converging to the root at $x = 0$:

n	x_n
0	1
1	-1
2	1
3	-1
4	1
5	-1
$\vdots$	$\vdots$



Diverging Approximations: Starting from $x_0 = 1.05$, the approximations diverge, moving farther and farther from the root at $x = 0$:

n	x_n
0	1.05
1	-1.211
2	2.061
3	-12.103
4	2653.453



These examples show two ways Newton's Method may fail. The approximations may oscillate or diverge, never converging to the root.