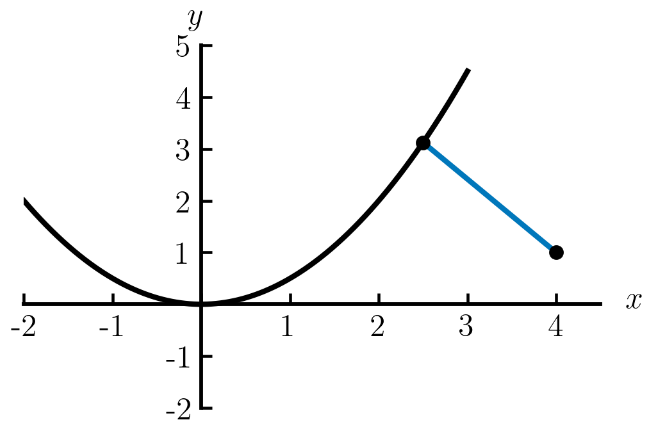
More Optimization

In this lesson, we will solve two more optimization problems focused on minimizing distance and time.

Example 1: Find the point on the parabola $y = \frac{1}{2}x^2$ that is closest to the point $(4, 1)$.



1) Label variables, identify constraints and objective:

Let (x, y) be a general point on the parabola.

Constraint: $y = \frac{1}{2}x^2$.

Objective: minimize the distance from (x, y) to $(4, 1)$, $\sqrt{(x-4)^2 + (y-1)^2}$.

2) Write the objective as a function of one variable:

Using the constraint $y = \dfrac{1}{2}x^2$, the distance is $\sqrt{(x-4)^2 + \left(\frac{1}{2}x^2 - 1\right)^2}$.

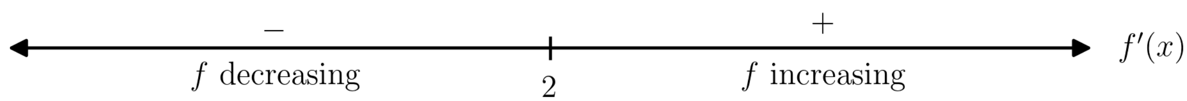
Minimizing distance is the same as minimizing the square of the distance, because the square root function is increasing. This is a common strategy when optimizing a distance.

$$f(x) = (x-4)^2 + \left(\frac{1}{2}x^2 - 1\right)^2.$$

3) Optimize the function: There are no restrictions on x, so we cannot use the Closed Interval Method. Instead, we can create a sign line for f'. Next we will find the derivative.

$$f'(x) = 2(x-4) + 2\left(\frac{1}{2}x^2 - 1\right) \cdot x = 2x - 8 + x^3 - 2x = x^3 - 8$$

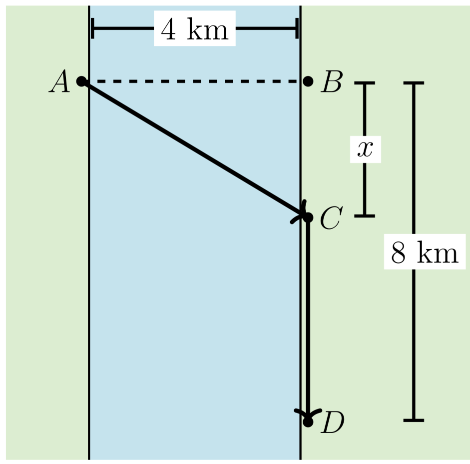
The derivative always exists, so the only critical point is where the derivative is 0. Solving $x^3 - 8 = 0$ gives $x = 2$. The point on the parabola is $\left(2, \frac{1}{2} \cdot 2^2\right) = (2, 2)$.



Since f is decreasing on $(-\infty,\, 2)$ and increasing on $(2,\, \infty)$, $x = 2$ gives the minimum value.

Therefore the point on $y = \frac{1}{2}x^2$ closest to $(4, 1)$ is $(2, 2)$.

Example 2: A person is at point A on the west side of a river, directly across from point B on the east side. The person wants to reach point D which is 8 km downriver from point B. The river is 4 km wide and straight. The person rows to point C between B and D, then runs the rest of the way to D. The person rows at 5 km/h and runs at 12 km/h. Assume the river current is negligible. Where should C be chosen to minimize the total time to reach D?



1) Label variables, identify constraints and objective:

Let x be the distance from B to C along the east bank, with $0 \leq x \leq 8$ km.

By the Pythagorean Theorem, the rowing distance is $\sqrt{4^2 + x^2} = \sqrt{16 + x^2}$ km.

The running distance from C to D is $(8 - x)$ km.

Objective: minimize total time to reach D.

2) Objective as a function of one variable:

The rowing time is the rowing distance $\sqrt{16 + x^2}$ divided by the rowing speed of 5 km/hr. The running time is the running distance $(8 - x)$ divided by the running speed of 12 km/hr. The total time is given by:

$$T(x) = \frac{\sqrt{16 + x^2}}{5} + \frac{8 - x}{12}$$

3) Optimize the function: We can use the Closed Interval Method to find the absolute minimum value of T on the interval $[0, 8]$. We can rewrite the objective function as

$$T(x) = \frac{1}{5}(16 + x^2)^{1/2} + \frac{1}{12}(8 - x).$$

$$\begin{aligned}
T(x) &= \frac{1}{5}(16 + x^2)^{1/2} + \frac{1}{12}(8 - x) \\
T'(x) &= \frac{1}{5} \cdot \frac{1}{2} \cdot (16 + x^2)^{-1/2} \cdot 2x - \frac{1}{12} \\
&= \frac{x}{5\sqrt{16 + x^2}} - \frac{1}{12} = 0 \\
\frac{x}{5\sqrt{16 + x^2}} &= \frac{1}{12} \\
12x &= 5\sqrt{16 + x^2} \\
144x^2 &= 25(16 + x^2) = 400 + 25x^2 \\
119x^2 &= 400 \\
x &= \frac{20}{\sqrt{119}} \approx 1.833 \quad (\text{since } x > 0)
\end{aligned}$$

The only critical point is $x = \dfrac{20}{\sqrt{119}} \approx 1.8334$ km, and we will also check the endpoints $x = 0$ and $x = 8$.

$$\begin{aligned}
T(0) &= \frac{\sqrt{16 + 0^2}}{5} + \frac{8 - 0}{12} = \frac{22}{15} \approx 1.4667 \\
T(8) &= \frac{\sqrt{16 + 8^2}}{5} + \frac{8 - 8}{12} = \frac{\sqrt{80}}{5} \approx 1.7889 \\
T\left(\frac{20}{\sqrt{119}}\right) &\approx 1.3939
\end{aligned}$$

The minimum travel time is approximately 1.3939 hours, which occurs when C is $\dfrac{20}{\sqrt{119}} \approx 1.8334$ km downriver from B.