Logarithmic Functions

In this lesson we will review logarithmic functions, their relationship with exponential functions, and the main logarithm properties.

If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is one-to-one, so it has an inverse function. The inverse is the **logarithmic function with base** b, written $\log_b$.

$$\log_b x = y \iff b^y = x$$

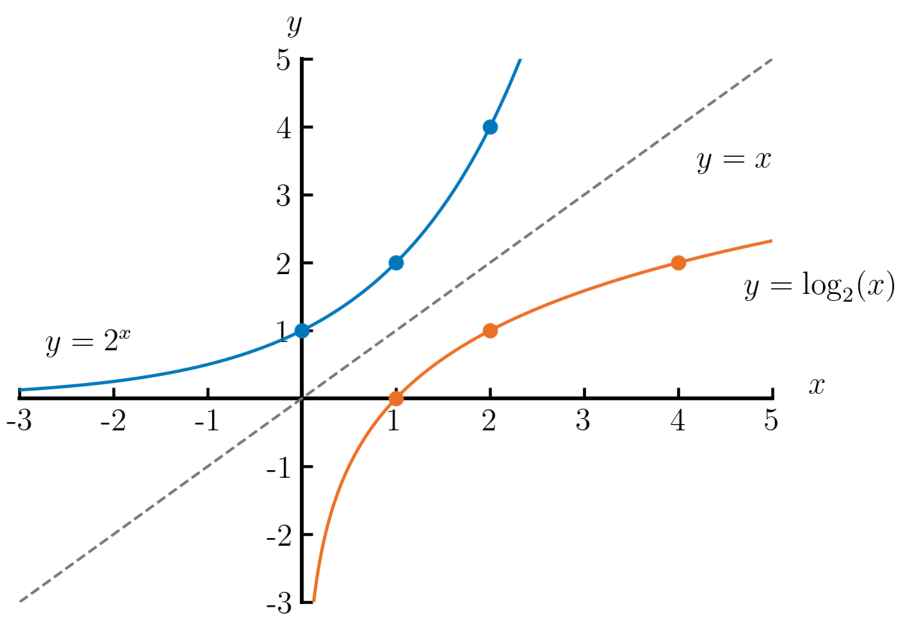
Example 1: Evaluate $\log_5(125)$.

Solution:

$\log_5(125) = y$ means $5^y = 125$. Since $5^3 = 125$, $\log_5(125) = 3$.

Example 2: Graph $y = 2^x$ and $y = \log_2(x)$ on the same axes. What is the relationship between these functions?

Solution:



The graphs are reflections across $y = x$ (inverse functions).

Cancellation Identities: Let $b > 0$ and $b \neq 1$. Since b^x and $\log_b x$ are inverses:

$$\begin{aligned} b^{\log_b(x)} &= x \qquad \text{for } x > 0 \\ \log_b(b^x) &= x \qquad \text{for all } x \end{aligned}$$

Properties of Logarithms: If x, y, and b are positive numbers, $b \neq 1$, and r is any real number,

- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
- $\log_b(x^r) = r \log_b x$

The inverse of the natural exponential function $f(x) = e^x$ is the **natural logarithm function**, written ln.

$$\log_e x = \ln x$$

$$\ln x = y \iff e^y = x$$

The cancellation identities become

$$\begin{aligned} e^{\ln(x)} &= x \qquad \text{for } x > 0 \\ \ln(e^x) &= x \qquad \text{for all } x \end{aligned}$$

Example 3: Solve for x: $\ln x = 5$

Solution:

Rewrite in exponential form $e^5 = x$, or exponentiate both sides:

$$\begin{aligned} \ln x &= 5 \\ e^{\ln x} &= e^5 \\ x &= e^5 \end{aligned}$$

Example 4: Solve the equation $e^{3x+2} = 7$.

Solution:

$$\begin{aligned} e^{3x+2} &= 7 \\ \ln\left(e^{3x+2}\right) &= \ln(7) \\ 3x + 2 &= \ln(7) \\ 3x &= \ln(7) - 2 \\ x &= \frac{\ln(7) - 2}{3} \end{aligned}$$

Change-of-base formulas: Let $b > 0$. Then

$$b^x = e^{\ln b^x} = e^{x \ln b}$$

For $b > 0$ with $b \neq 1$ and $x > 0$,

$$\log_b x = \frac{\ln x}{\ln b}$$

Proof: Let $y = \log_b x$. In exponential form: $b^y = x$

$$\begin{aligned} \ln(b^y) &= \ln(x) \\ y \ln b &= \ln x \\ y &= \frac{\ln x}{\ln b} \\ \log_b x &= \frac{\ln x}{\ln b} \end{aligned}$$

Example 5: How can we evaluate $\log_7(23)$, only using a calculator's ln button?

Solution:

$$\log_7(23) = \frac{\ln(23)}{\ln(7)}$$