Logarithmic Differentiation

We will use logarithmic differentiation to make certain derivatives easier to find, and find derivatives of functions of the form $y = f(x)^{g(x)}$.

We have all the tools we need to find derivatives of functions like $y = \dfrac{(x^3 - x - 2)\sqrt{x^2+1}}{(x+2)^5}$, but it would be tedious to use the product, quotient, and chain rules directly. Instead, we can use logarithmic differentiation. We start by taking the natural log on both sides and using the log properties to expand the right-hand side as much as possible. Then, we use implicit differentiation to find $\dfrac{dy}{dx}$.

Example 1: Find the derivative of $y = \dfrac{(x^3 - x - 2)\sqrt{x^2+1}}{(x+2)^5}$.

Solution: Start by taking the natural log on both sides, and expanding the right-hand side:

$$\begin{aligned}
\ln y &= \ln\left(\frac{(x^3 - x - 2)\sqrt{x^2+1}}{(x+2)^5}\right) \\
\ln y &= \ln(x^3 - x - 2) + \ln\sqrt{x^2+1} - \ln(x+2)^5 \\
\ln y &= \ln(x^3 - x - 2) + \frac{1}{2}\ln(x^2+1) - 5\ln(x+2)
\end{aligned}$$

Now we can use implicit differentiation to find $\dfrac{dy}{dx}$. Differentiating both sides with respect to x, we get:

$$\begin{aligned}
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{x^3 - x - 2}\cdot(3x^2 - 1) + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\cdot\frac{1}{x+2}\cdot 1 \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{3x^2 - 1}{x^3 - x - 2} + \frac{x}{x^2+1} - \frac{5}{x+2} \\
\frac{dy}{dx} &= y\left(\frac{3x^2 - 1}{x^3 - x - 2} + \frac{x}{x^2+1} - \frac{5}{x+2}\right) \\
\frac{dy}{dx} &= \frac{(x^3 - x - 2)\sqrt{x^2+1}}{(x+2)^5}\left(\frac{3x^2 - 1}{x^3 - x - 2} + \frac{x}{x^2+1} - \frac{5}{x+2}\right)
\end{aligned}$$

Example 2: Find the derivative of $y = \dfrac{\sqrt[3]{2x-4}}{x^5(3x-1)^2}$.

Solution: We will use logarithmic differentiation. Start by taking the natural log on both sides and expanding:

$$\begin{aligned}
\ln y &= \ln\Big(\frac{\sqrt[3]{2x-4}}{x^5(3x-1)^2}\Big) \\
\ln y &= \ln \sqrt[3]{2x-4} - \ln x^5 - \ln(3x-1)^2 \\
\ln y &= \frac{1}{3}\ln(2x-4) - 5\ln x - 2\ln(3x-1) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{3}\cdot\frac{1}{2x-4}\cdot 2 - 5\cdot\frac{1}{x} - 2\cdot\frac{1}{3x-1}\cdot 3 \\
\frac{dy}{dx} &= y\Big(\frac{2}{3(2x-4)} - \frac{5}{x} - \frac{6}{3x-1}\Big) \\
\frac{dy}{dx} &= \frac{\sqrt[3]{2x-4}}{x^5(3x-1)^2}\Big(\frac{2}{3(2x-4)} - \frac{5}{x} - \frac{6}{3x-1}\Big)
\end{aligned}$$

We can also find derivatives of functions of the form $y = f(x)^{g(x)}$ using logarithmic differentiation.

Example 3: Find the derivative $\dfrac{d}{dx}(x^x)$.

Solution: We will start by taking the natural log on both sides:

$$\begin{aligned}
\ln y &= \ln(x^x) \\
\ln y &= x\ln x \\
\frac{1}{y}\cdot\frac{dy}{dx} &= x\cdot\frac{d}{dx}(\ln x) + \ln x\cdot\frac{d}{dx}(x) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= x\cdot\frac{1}{x} + \ln x\cdot 1 \\
\frac{1}{y}\cdot\frac{dy}{dx} &= 1 + \ln x \\
\frac{dy}{dx} &= y(1+\ln x) = x^x(1+\ln x)
\end{aligned}$$