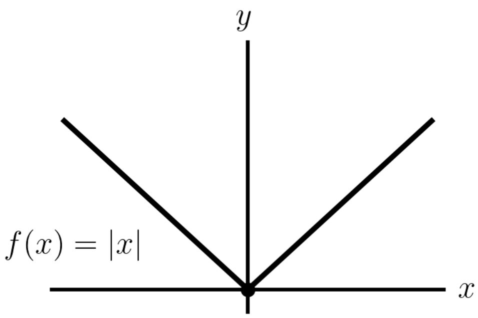
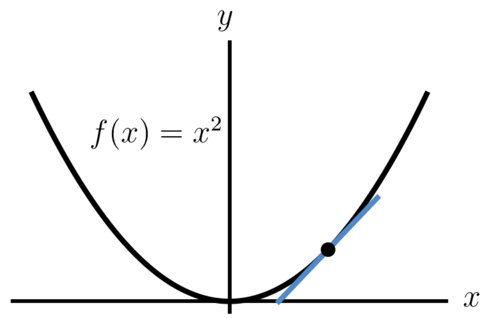
Linear Approximations

In this lesson, we will find linear approximations of functions and use them to estimate values.

If f is differentiable at a, then f is locally linear at a. If we zoom in on the graph of f at a, the graph looks more and more like a line.



$f(x) = |x|$ is not differentiable at 0. No matter how much we zoom in, the graph always has a sharp corner at 0.



$f(x) = x^2$ is differentiable at 1. When we zoom in, the graph looks like its tangent line.

If $y = f(x)$ is differentiable at a, then an equation for the tangent line at a is

$$\begin{aligned} y - f(a) &= f'(a)(x-a) \\ y &= f(a) + f'(a)(x-a) \end{aligned}$$

Definition Linear Approximation: Suppose f is differentiable at a. The linear approximation (or linearization) of f at a is

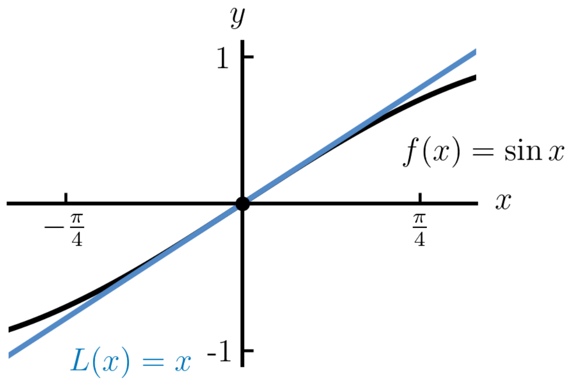
$$L(x) = f(a) + f'(a)(x-a).$$

Example 1: Find the linear approximation to $f(x) = \sin x$ at $a = 0$. Then use it to estimate $\sin(0.15)$. How does this compare to the exact value?

Solution: We have $f(x) = \sin x$ and $f'(x) = \cos x$. Evaluating these at $a = 0$, we get $f(0) = 0$ and $f'(0) = 1$. The linear approximation is given by

$$L(x) = f(0) + f'(0)(x - 0) = 0 + 1 \cdot (x - 0) = x.$$

Therefore $\sin(0.15) \approx L(0.15) = 0.15$. The true value is $\sin(0.15) \approx 0.149438$. The approximation is close, with an error of about $0.15 - 0.149438 = 0.000562$.



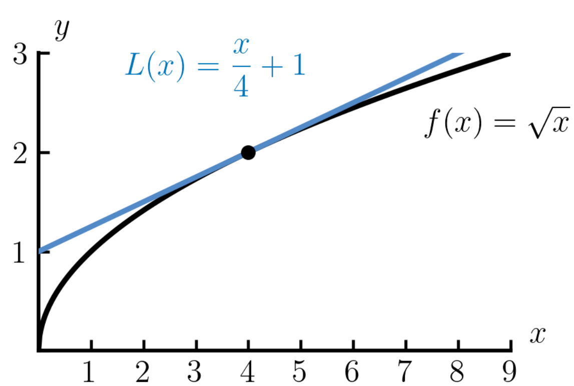
The linear approximation shows that $\sin x \approx x$ when x is near 0, and x is measured in radians.

Example 2: Use a linear approximation to estimate $\sqrt{5}$. Is the approximation an overestimate or an underestimate?

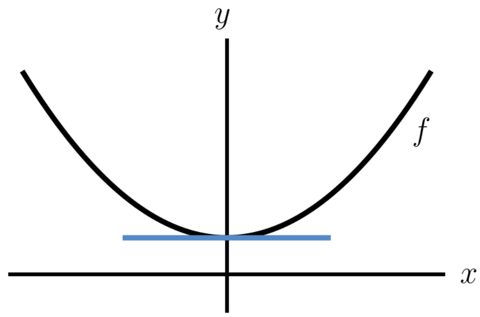
Solution: We want to find a linear approximation to $f(x) = \sqrt{x} = x^{1/2}$ at a point near 5. The closest perfect square is 4, so we can use $a = 4$. Then $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. Evaluating at $a = 4$, we get $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation is given by

$$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) = \frac{x}{4} + 1.$$

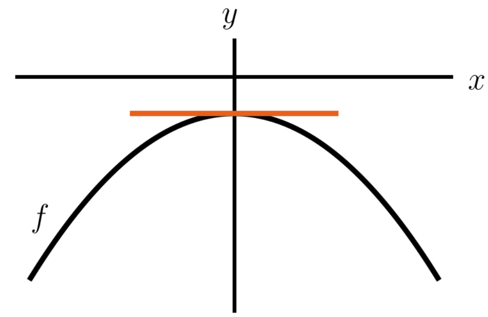
We approximate $\sqrt{5} \approx L(5) = \frac{5}{4} + 1 = 2.25$. The exact value is $\sqrt{5} \approx 2.23607$, so our approximation is an overestimate. Since f is concave down, the tangent line lies above the graph.



Linear Approximation and Concavity: The concavity of f determines whether the tangent line lies above or below the graph.



Concave up: tangent line lies below the graph, so $L(x)$ underestimates $f(x)$.



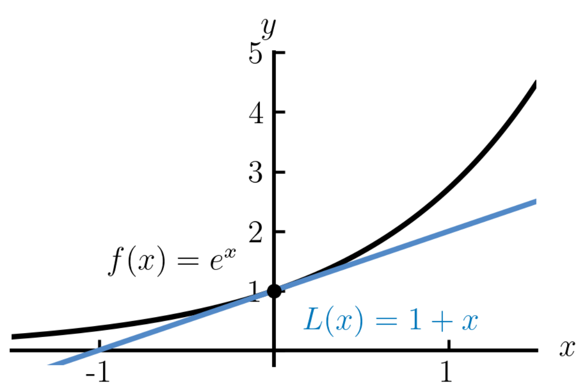
Concave down: tangent line lies above the graph, so $L(x)$ overestimates $f(x)$.

Example 3: Use a linear approximation to estimate $e^{0.04}$. Is the approximation an overestimate or an underestimate?

Solution: We want to find a linear approximation to $f(x) = e^x$ at a point near 0.04. A nearby point we can evaluate exactly is $a = 0$. Then $f'(x) = e^x$. Evaluating at $a = 0$, we get $f(0) = 1$ and $f'(0) = 1$. The linear approximation is given by

$$L(x) = f(0) + f'(0)(x - 0) = 1 + 1(x - 0) = 1 + x.$$

We approximate $e^{0.04} \approx L(0.04) = 1 + 0.04 = 1.04$. The exact value is $e^{0.04} \approx 1.04081$. The approximation is an underestimate, since f is concave up.



The linear approximation shows that $e^x \approx 1 + x$ when x is near 0.