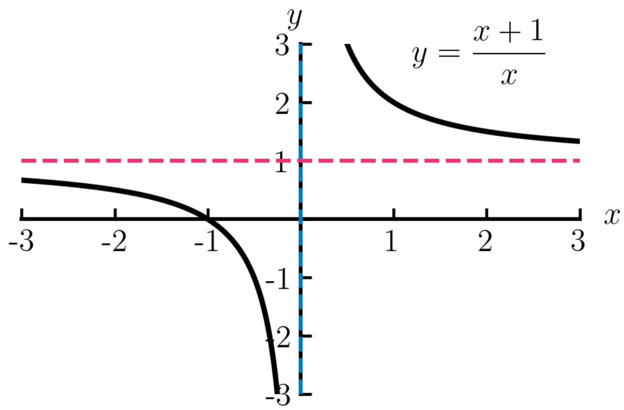
Limits at Infinity

In this lesson, we will find limits at infinity and explore the end behavior of exponential and logarithmic functions.

What happens to $f(x) = \dfrac{x+1}{x}$ as x gets larger and larger?



x	$f(x)$
10	1.1
100	1.01
1000	1.001

Definition Limit at Infinity: Let f be defined for sufficiently large x. We say $\lim_{x\to\infty} f(x) = L$, if we can make $f(x)$ as close as we want to L by choosing x sufficiently large.

Read Aloud As: "The limit of $f(x)$ as x approaches infinity equals L."

We can define $\lim_{x\to-\infty} f(x) = L$ similarly. The limits of a function at $\pm\infty$ determine its **end behavior**.

Definition Horizontal Asymptote: The line $y = L$ is called a horizontal asymptote of f if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Example 1: Evaluate $\lim_{x\to\infty} \left(5 - \dfrac{3}{x}\right)$.

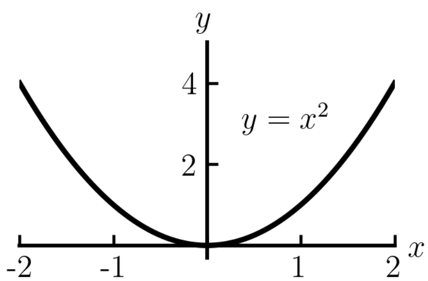
Solution:

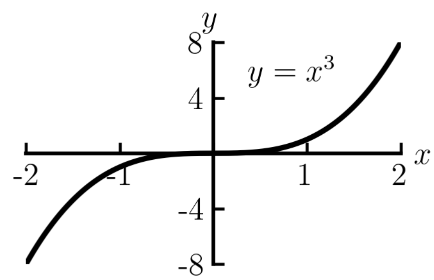
As $x \to \infty$, the denominator of $\dfrac{3}{x}$ gets larger and larger. So $\dfrac{3}{x} \to 0$. Then, $\lim_{x\to\infty} \left(5 - \dfrac{3}{x}\right) = 5 - 0 = 5$.

Definition Infinite Limit at Infinity: Let f be defined for sufficiently large x. If we can make $f(x)$ as large as we want by choosing x sufficiently large, we say $\lim_{x\to\infty} f(x) = \infty$. If we can make $f(x)$ negative and as large in magnitude as we want by choosing x sufficiently large, we say $\lim_{x\to\infty} f(x) = -\infty$.

Read Aloud As: "The limit of $f(x)$ as x approaches infinity is infinity," or "The limit of $f(x)$ as x approaches infinity is negative infinity."

We can define $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ similarly.





Theorem Limit at Infinity of x^n: Let n be a positive integer. Then:

- $\lim_{x\to\pm\infty} x^{-n} = \lim_{x\to\pm\infty} \frac{1}{x^n} = 0$.
- If n is even, $\lim_{x\to\pm\infty} x^n = \infty$.
- If n is odd, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$.

Example 2: Evaluate $\lim_{x\to\infty} (x^3 - x^2 + 4x + 3)$.

Solution:

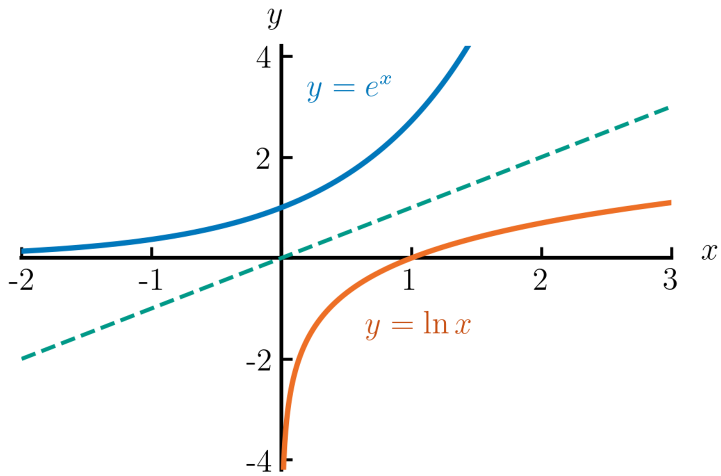
Factor out the highest power: $x^3 - x^2 + 4x + 3 = x^3\left(1 - \frac{1}{x} + \frac{4}{x^2} + \frac{3}{x^3}\right)$. As $x \to \infty$, the factor in parentheses approaches $(1 - 0 + 0 + 0) = 1$, and $x^3 \to \infty$.

$$\lim_{x\to\infty} (x^3 - x^2 + 4x + 3) = \infty$$

The end behavior of a polynomial is determined by the highest order term.

Example 3: Find $\lim_{x\to-\infty} e^x$ and $\lim_{x\to 0^+} \ln x$.

Solution:



$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to 0^+} \ln x = -\infty.$$

Theorem End Behavior of Exponential and Logarithmic Functions:

- $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.
- $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to\infty} \ln x = \infty$.
- $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to-\infty} e^{-x} = \infty$.

This theorem is also true if we replace e with any base $b > 1$.

Example 4: Find $\lim_{x\to 0^+} (3x - \ln x)$.

Solution:

As $x \to 0^+$, $\ln x \to -\infty$, so $-\ln x \to \infty$. Also $3x \to 0$.

$$\lim_{x\to 0^+} (3x - \ln x) = \infty$$

Example 5: Find $\lim_{x \to \infty} \sin x$.

Solution: As $x \to \infty$, $\sin x$ oscillates between -1 and 1 and does not approach a single value. The limit does not exist.