Initial Value Problems

In this lesson, we will explore initial value problems, find position functions from velocity, and find velocity functions from acceleration.

In the previous lesson, we found general antiderivatives: entire families of antiderivatives of the form $F(x)+C$. But sometimes we want to obtain a specific antiderivative that satisfies an initial condition $F(a)=b$. This is called an **initial value problem**.

Example 1: Let $f(x) = \sqrt{x} + \frac{1}{x}$. Find the antiderivative of f satisfying the initial condition $F(1) = 3$.

Solution: To find the general antiderivative, we use the linearity of integration and the power rule:

$$\begin{aligned}\int \left(\sqrt{x} + \frac{1}{x}\right) dx &= \int \sqrt{x}\, dx + \int \frac{1}{x}\, dx \\ &= \int x^{1/2}\, dx + \int x^{-1}\, dx \\ &= \frac{2}{3}x^{3/2} + \ln|x| + C\end{aligned}$$

Next, we use the initial condition $F(1) = 3$ to find the constant C:

$$\begin{gathered}F(1) = \frac{2}{3}(1)^{3/2} + \ln|1| + C = 3 \\ \frac{2}{3} + C = 3 \quad C = \frac{7}{3}\end{gathered}$$

The antiderivative satisfying the initial condition is

$$F(x) = \frac{2}{3}x^{3/2} + \ln|x| + \frac{7}{3}.$$

One-Dimensional Motion: Let $s(t)$ be the position of an object at time t. We have seen that

$$\begin{aligned}\text{velocity} &= v(t) = s'(t) \\ \text{acceleration} &= a(t) = v'(t)\end{aligned}$$

Sometimes we know the acceleration or velocity but not the position function. We can solve for position from velocity, by solving the initial value problem:

$$s'(t) = v(t), \quad s(0) = s_0$$

We can solve for velocity from acceleration, by solving the initial value problem:

$$v'(t) = a(t), \quad v(0) = v_0$$

Example 2: A ball is thrown upwards from the edge of a 50 m tall cliff, with initial velocity 12 m/s. The acceleration from gravity is approximately $a(t) = -9.8\,\text{m/s}^2$. Find the position function for the ball. What is the maximum height the ball reaches?

Solution: Say the ball is thrown at time $t = 0$. Then the initial position is $s(0) = 50$ m and the initial velocity is $v(0) = 12$ m/s. To find the velocity, we find the antiderivative of acceleration:

$$\begin{aligned}\int a(t)\,dt = \int -9.8\,dt &= -9.8t + C \\ 12 = v(0) &= -9.8(0) + C, \text{so } C = 12 \\ v(t) &= -9.8t + 12\end{aligned}$$

To find the position, we find the antiderivative of velocity:

$$\begin{aligned}\int v(t)\,dt = \int (-9.8t + 12)\,dt &= -4.9t^2 + 12t + C \\ 50 = s(0) &= -4.9(0)^2 + 12(0) + C, \text{so } C = 50 \\ s(t) &= -4.9t^2 + 12t + 50\end{aligned}$$

The ball reaches its maximum height when the velocity is 0:

$$\begin{aligned}v(t) &= 0 \\ -9.8t + 12 &= 0 \\ t &= \frac{12}{9.8} \approx 1.2245 \text{ s} \\ s(1.2245) &= -4.9(1.2245)^2 + 12(1.2245) + 50 \approx 57.35 \text{ m}\end{aligned}$$

The position function is $s(t) = -4.9t^2 + 12t + 50$, and the maximum height is approximately 57.35 m.

Example 3: At time $t = 0$, a car begins accelerating from rest, with acceleration given by $a(t) = 30t - 12t^2$. How far does the car travel in the first 2 seconds?

Solution: Since the car starts from rest, the initial velocity is $v(0) = 0$ and say the initial position is $s(0) = 0$. To find the velocity, we find the antiderivative of acceleration:

$$\begin{aligned} \int a(t)\,dt &= \int (30t - 12t^2)\,dt = 15t^2 - 4t^3 + C \\ 0 &= v(0) = 15(0)^2 - 4(0)^3 + C, \text{so } C = 0 \\ v(t) &= 15t^2 - 4t^3 \end{aligned}$$

To find the position, we find the antiderivative of velocity:

$$\begin{aligned} \int v(t)\,dt &= \int (15t^2 - 4t^3)\,dt = 5t^3 - t^4 + C \\ 0 &= s(0) = 5(0)^3 - (0)^4 + C, \text{so } C = 0 \\ s(t) &= 5t^3 - t^4 \end{aligned}$$

The distance traveled in the first 2 seconds is:

$$s(2) - s(0) = (5(2)^3 - (2)^4) - (5(0)^3 - (0)^4) = 40 - 16 = 24 \text{ m}$$

The car travels 24 m in the first 2 seconds.