Implicit Differentiation

We will explore implicit functions, and use implicit differentiation to find their derivatives.

So far we have found derivatives of explicit functions, $y = f(x)$. But sometimes we only have an equation relating x and y, like $x^2 + y^2 = 1$. We say the equation implicitly defines one or more functions. In some cases, we can solve the equation for y to get an explicit function.

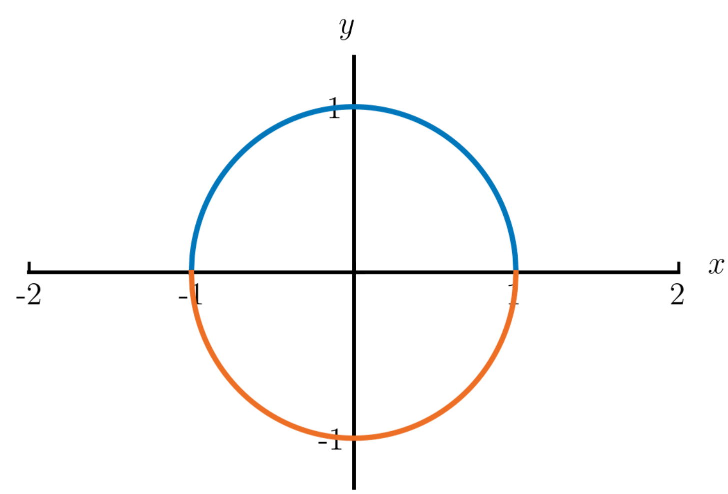
Example 1: Solve $x^2 + y^2 = 1$ for y to write the implicit functions in explicit form.

Solution: Solving for y, we get $y^2 = 1 - x^2$, and then

$$y = \pm\sqrt{1 - x^2}$$

So there are two functions implicitly defined by the original equation:

$$y = \sqrt{1 - x^2} \quad \text{and} \quad y = -\sqrt{1 - x^2}$$



Sometimes it is impossible or very difficult to solve for y explicitly. Luckily, we can find derivatives implicitly, by viewing y as a function of x and applying the chain rule. Then we will solve the equation for $\frac{dy}{dx}$.

Example 2: Use implicit differentiation to find $\frac{dy}{dx}$ for $x^2 + y^2 = 1$ and find an equation of the tangent line to the circle at the point $\left(\frac{4}{5}, \frac{3}{5}\right)$.

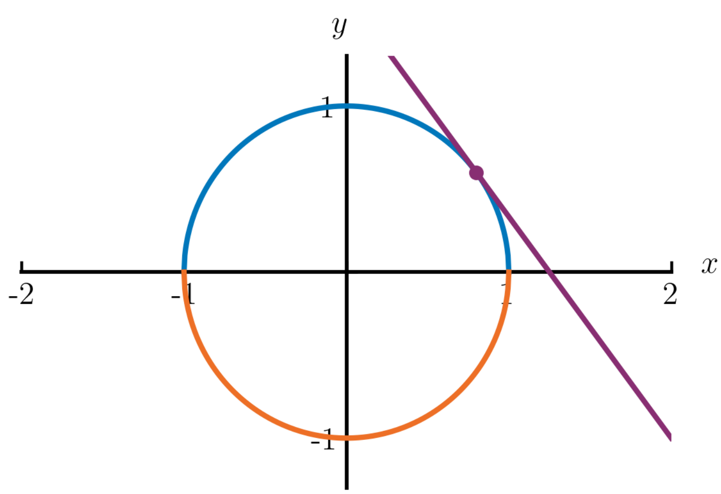
Solution: We will differentiate both sides of the equation with respect to x. Note that y is a function of x, so we need to apply the chain rule to find $\frac{d}{dx}(y^2) = 2y \cdot \frac{d}{dx}(y) = 2y\frac{dy}{dx}$. Differentiating both sides of the equation, we get:

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we get $2y\frac{dy}{dx} = -2x$, so $\frac{dy}{dx} = -\frac{x}{y}$. At the point $\left(\frac{4}{5}, \frac{3}{5}\right)$, the slope of the tangent line is

$$\frac{dy}{dx} = -\frac{\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

An equation of the tangent line is $y - \frac{3}{5} = -\frac{4}{3}\left(x - \frac{4}{5}\right)$. We can simplify this to $y = -\frac{4}{3}x + \frac{5}{3}$.



Example 3: Find $\dfrac{dy}{dx}$ for $xe^y = y + 1$.

Solution: Differentiating both sides, we get:

$$\begin{aligned} x \cdot \frac{d}{dx}(e^y) + e^y \cdot \frac{d}{dx}(x) &= \frac{d}{dx}(y+1) \\ xe^y \frac{dy}{dx} + e^y \cdot 1 &= \frac{dy}{dx} \end{aligned}$$

Next move the terms involving $\dfrac{dy}{dx}$ to one side and factor:

$$\begin{aligned} xe^y \frac{dy}{dx} - \frac{dy}{dx} &= -e^y \\ (xe^y - 1) \cdot \frac{dy}{dx} &= -e^y \\ \frac{dy}{dx} &= -\frac{e^y}{xe^y - 1} \end{aligned}$$

Example 4: Find an equation of the tangent line to $\tan(xy) = x + y$ at the point $(0, 0)$.

Solution: Differentiate both sides:

$$\begin{aligned} \frac{d}{dx}(\tan(xy)) &= \frac{d}{dx}(x+y) \\ \sec^2(xy) \cdot \frac{d}{dx}(xy) &= 1 + \frac{dy}{dx} \\ \sec^2(xy) \cdot \left(x \cdot \frac{dy}{dx} + y \cdot 1 \right) &= 1 + \frac{dy}{dx} \\ x \sec^2(xy) \frac{dy}{dx} + y \sec^2(xy) &= 1 + \frac{dy}{dx} \\ x \sec^2(xy) \frac{dy}{dx} - \frac{dy}{dx} &= 1 - y \sec^2(xy) \\ (x \sec^2(xy) - 1) \frac{dy}{dx} &= 1 - y \sec^2(xy) \\ \frac{dy}{dx} &= \frac{1 - y \sec^2(xy)}{x \sec^2(xy) - 1} \end{aligned}$$

Substituting $x = 0$ and $y = 0$, we get $\dfrac{dy}{dx} = \dfrac{1-0}{0-1} = -1$. An equation of the tangent line is $y = -x$.

Example 5: Find the second derivative $\dfrac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$.

Solution: In Example 2, we found $\dfrac{dy}{dx} = -\dfrac{x}{y}$. We can find the second derivative by differentiating both sides of this equation with respect to x.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x}{y}\right) = -\frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

Substituting $\dfrac{dy}{dx} = -\dfrac{x}{y}$, we get:

$$\frac{d^2y}{dx^2} = -\frac{y + x \cdot \frac{x}{y}}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3}.$$

In the last step, we used the original equation $x^2 + y^2 = 1$ to simplify the expression.