Exponential Functions

In this lesson we will review exponential functions and explore the number e.

Exponential functions have the form $f(x) = b^x$ for some positive constant b.

For what values of x can we evaluate b^x?

- If $x = n$, a positive integer, then $b^n = \underbrace{b \cdot b \cdot b \cdots b}_{n \text{ times}}$
- If $x = 0$, then $b^0 = 1$
- If $x = -n$, then $b^{-n} = \dfrac{1}{b^n}$
- If x is a rational number, $x = \dfrac{p}{q}$, where p and q are integers and $q > 0$, then

$$b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$$

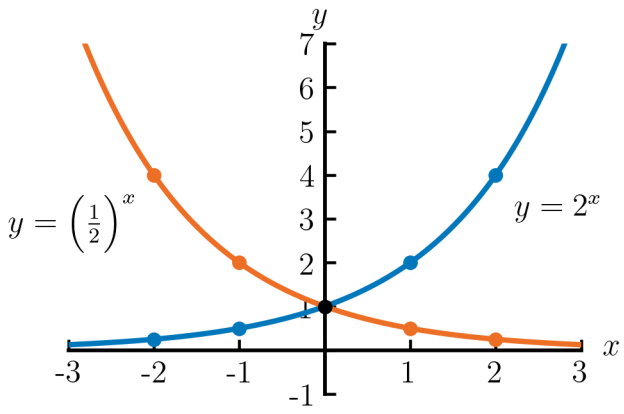
- What do we do if x is an irrational number?

Say we want to find 2^π. We can approximate $3 < \pi < 4$, so $2^3 < 2^\pi < 2^4$. To get a better approximation, we use a better approximation of π:

$$2^{3.1} < 2^\pi < 2^{3.2}$$
$$2^{3.14} < 2^\pi < 2^{3.15}$$
$$\vdots$$

Example 1: Sketch the graphs of $y = 2^x$ and $y = \left(\dfrac{1}{2}\right)^x$. How are the graphs related?

Solution:



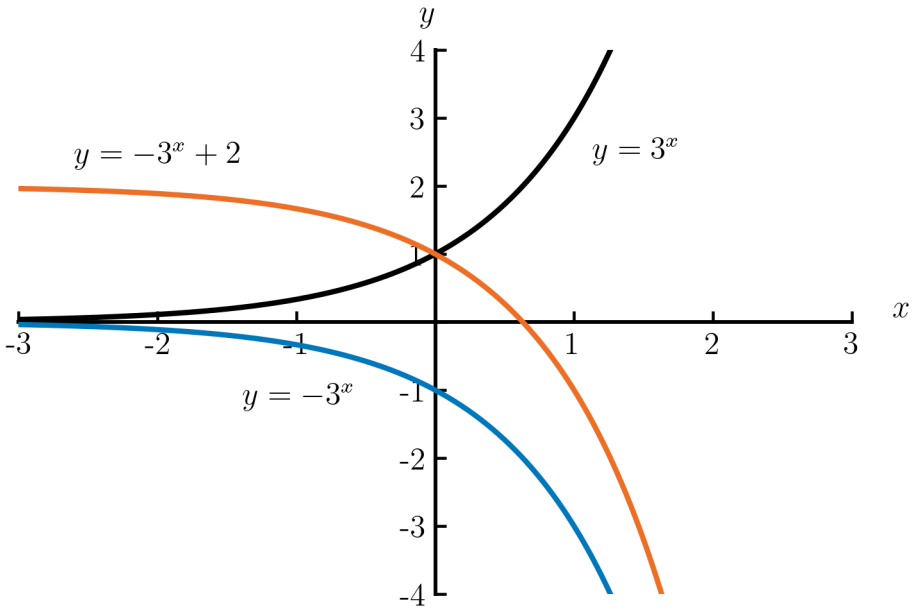
Notice $y = \left(\dfrac{1}{2}\right)^x = \dfrac{1}{2^x} = 2^{-x}$. The graph of $y = 2^{-x}$ is a reflection of $y = 2^x$ across the y-axis.

Example 2: Sketch the graph of $f(x) = -3^x + 2$. What is its domain and range?

Solution:

Start with $y = 3^x$. Then

- Reflect across the x-axis to get $y = -3^x$
- Shift up 2 to get $y = -3^x + 2$

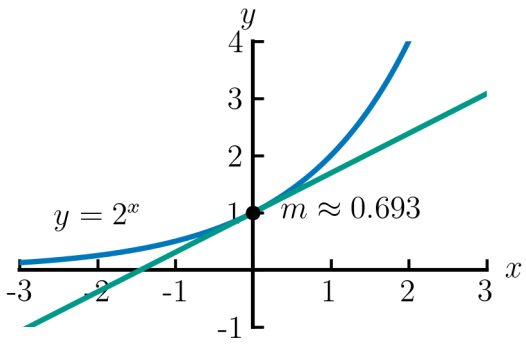


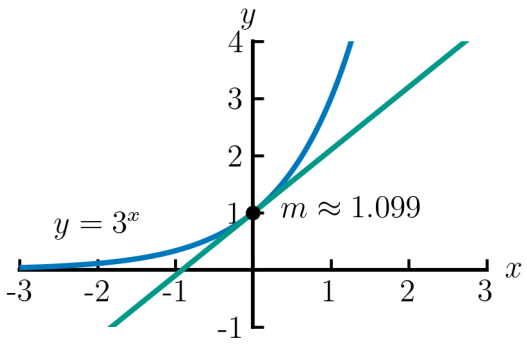
Domain: All real numbers, $(-\infty, \infty)$ Range: $(-\infty, 2)$

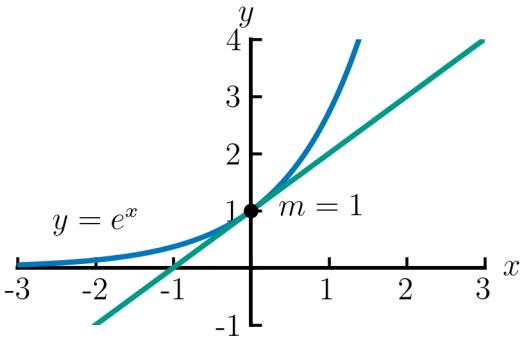
Exponent Rules: If a and b are positive real numbers, and x and y are any real numbers:

$$\begin{aligned} b^{x+y} &= b^x b^y \\ b^{x-y} &= \frac{b^x}{b^y} \\ (b^x)^y &= b^{xy} \\ (ab)^x &= a^x b^x \end{aligned}$$

The Number e: e is an irrational number which is approximately $e \approx 2.71828$. The function $f(x) = e^x$ is called the **natural exponential function** and has an important property for calculus.







If we use e as the base of the exponential function, then the line tangent to the graph of $y = e^x$ when $x = 0$ has slope exactly 1.