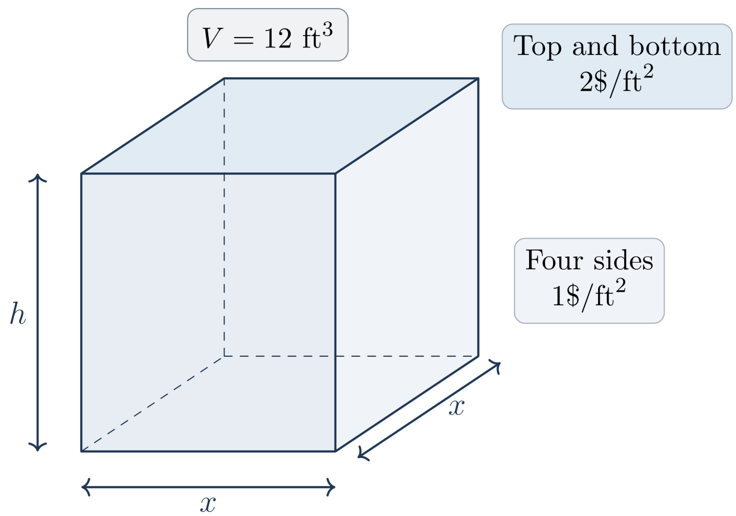
Even More Optimization

In this lesson, we will solve one more optimization problem about minimizing cost.

Example 1: A product designer is designing a box with volume 12 ft^3. The base of the box must be square. The material for the top and bottom costs \$2 per ft^2, and the material for the other four sides costs \$1 per ft^2. What dimensions minimize the total cost of the material?



1) Label variables, identify constraints and objective:

Let the square base have side length x and let the height be h.

Constraint (volume): $V = x^2h = 12$.

The top and bottom each have area x^2, so the combined area is $2x^2$. The cost for the top and bottom is $2(2x^2) = 4x^2$ dollars.

The four sides each have area xh, for a total area of $4xh$. The cost for the sides is $4xh$ dollars.

Objective: minimize total cost $C = 4x^2 + 4xh$.

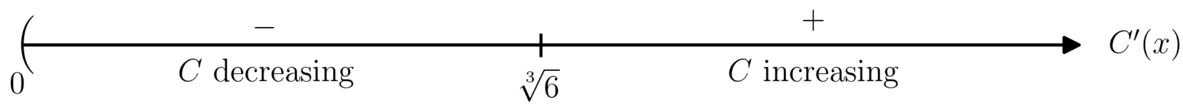
2) Write the cost as a function of one variable:

From the constraint $V = x^2h = 12$, we can solve for h: $h = \dfrac{12}{x^2}$. Substituting this into the cost function, we get:

$$C(x) = 4x^2 + 4x\left(\frac{12}{x^2}\right) = 4x^2 + \frac{48}{x}.$$

3) Optimize the function: We know $x \geq 0$, but there is no upper bound on x, so we cannot use the Closed Interval Method. Instead, we can create a sign line for C'.

$$
\begin{aligned}
C'(x) &= 8x - \frac{48}{x^2} = 0 \\
8x^3 - 48 &= 0 \\
8x^3 &= 48 \\
x^3 &= 6 \quad \text{so} \quad x = \sqrt[3]{6}
\end{aligned}
$$



Since C is decreasing until $x = \sqrt[3]{6}$ and then increasing, $x = \sqrt[3]{6} \approx 1.8171$ gives the minimum cost.

Then $h = \dfrac{12}{x^2} = \dfrac{12}{6^{2/3}} \approx 3.6342$ feet.

So the dimensions that minimize cost are a square base side length $x \approx 1.8171$ ft and height $h \approx 3.6342$ ft.