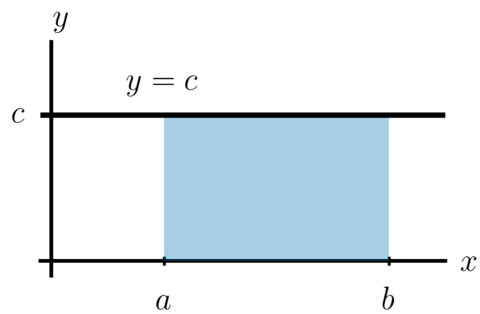
Evaluating Definite Integrals using Graphs and Geometry

In this lesson, we will use graphs and geometry to evaluate definite integrals.

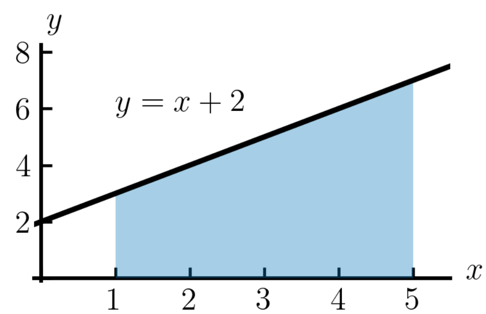
Say we want to evaluate $\int_a^b c\,dx$, where c is a constant. The definite integral gives the net area bounded by the graph of $y = c$ and the x-axis.



The net area is a rectangle with width $b - a$ and height c. The area is $c(b - a)$. So,

$$\boxed{\int_a^b c\,dx = c(b - a)}$$

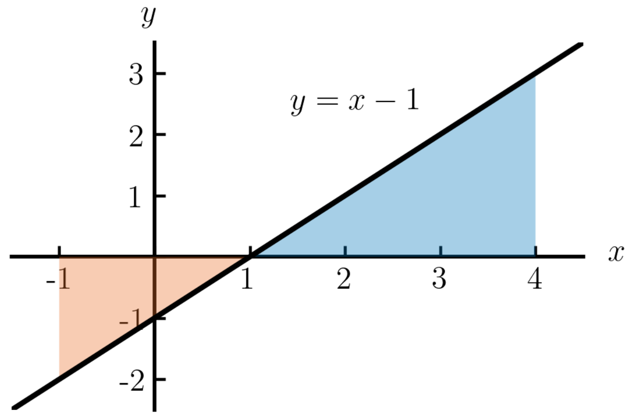
Example 1: Use geometry to evaluate $\int_1^5 (x + 2)\,dx$.



Solution: The region under $y = x + 2$ on $[1, 5]$ is a trapezoid. The area of a trapezoid is $\frac{1}{2}(b_1 + b_2)h$. This trapezoid is sideways, so the height runs along the x-axis: $h = 5 - 1 = 4$. The bases are the vertical lengths $1 + 2 = 3$ and $5 + 2 = 7$. So,

$$\int_1^5 (x + 2)\,dx = \frac{1}{2}(3 + 7)(4) = \boxed{20}$$

Example 2: Use geometry to evaluate $\int_{-1}^{4} (x - 1)\, dx$.



Solution: The definite integral gives the net area: the area of the triangle above the x-axis minus the area of the triangle below the x-axis.

The triangle above the x-axis has base $4 - 1 = 3$ and height 3, so the area is $\frac{1}{2}(3)(3) = \frac{9}{2}$.

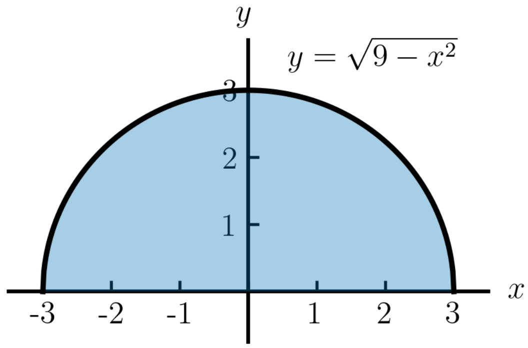
The triangle below the x-axis has base $1 - (-1) = 2$ and height 2, so the area is $\frac{1}{2}(2)(2) = 2$.

The net area is $\frac{9}{2} - 2 = \frac{5}{2}$. So,

$$\int_{-1}^{4} (x - 1)\, dx = \boxed{\frac{5}{2}}$$

Now try these interactive problems on the website, to practice some new cases: an area under a curved function and using a graph with labeled areas to find a definite integral.

Interactive Problem 1: Use geometry to evaluate $\int_{-3}^{3} \sqrt{9-x^2}\, dx$.



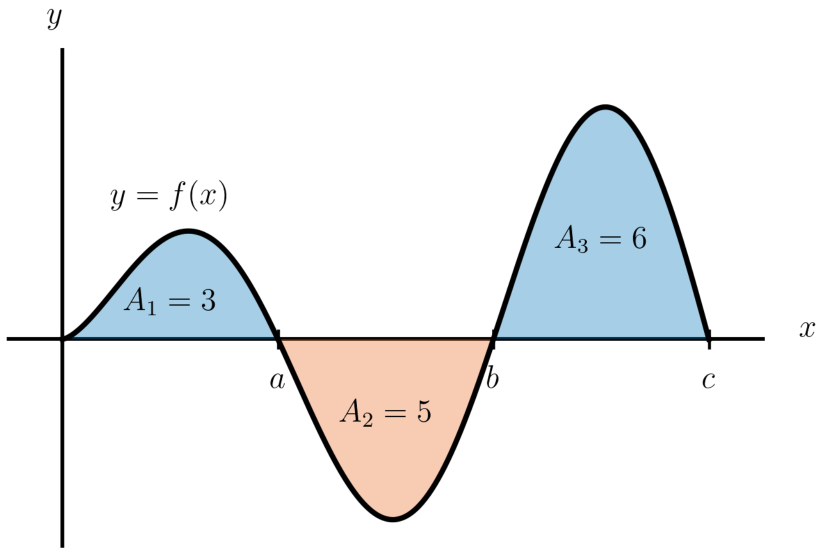
(a) On the graph, the definite integral $\int_{-3}^{3} \sqrt{9-x^2}\, dx$ is the area of the semicircle.

(b) The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

(c) The radius of the semicircle is 3.

(d) Putting this all together, the definite integral is $\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{1}{2}\pi(3)^2 = \frac{9}{2}\pi$.

Interactive Problem 2: The graph shows the areas $A_1 = 3$, $A_2 = 5$, and $A_3 = 6$ of regions bounded by the graph of $y = f(x)$ and the x-axis. We will use the graph to find $\int_0^b f(x)\,dx$ and $\int_a^c f(x)\,dx$.



(a) The areas involved in finding $\int_0^b f(x)\,dx$ are A_1 and A_2.

(b) A_1 counts as positive area.

(c) A_2 counts as negative area.

(d) So, $\int_0^b f(x)\,dx = A_1 - A_2 = 3 - 5 = -2$.

(e) The areas involved in finding $\int_a^c f(x)\,dx$ are A_2 and A_3.

(f) The net area is $\int_a^c f(x)\,dx = -A_2 + A_3 = -5 + 6 = 1$.