Differentials

In this lesson, we will introduce differentials, and use them to approximate changes in functions.

If $y = f(x)$ is differentiable at a, the linear approximation of f at a is

$$f(x) \approx f(a) + f'(a)(x - a).$$

Let $\Delta x = x - a$ denote the change in x, and let $\Delta y = f(x) - f(a)$ denote the corresponding change in $y = f(x)$.

$$f(x) - f(a) \approx f'(a)(x - a)$$
$$\Delta y \approx f'(a)\Delta x$$

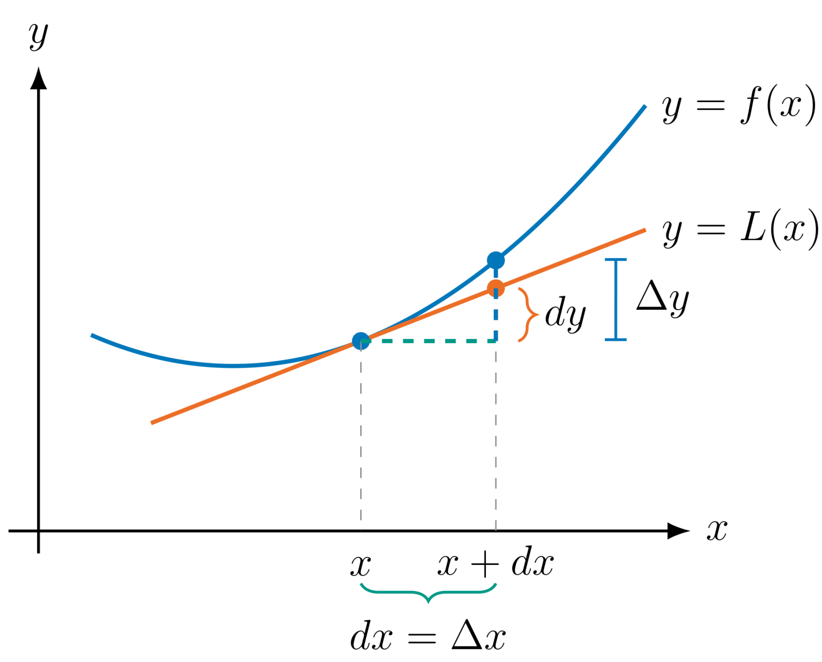
In the limit as $\Delta x \to 0$, this approximation gets better and better, and in the limit is exactly equal. This is called the differential form:

$$dy = f'(a)\,dx.$$

Now we will replace the fixed point a with the variable x. If f is differentiable at x, then

$$dy = f'(x)\,dx.$$

The quantities dx and dy are called **differentials**.



When x changes by $dx = \Delta x$,

- Actual change: $\Delta y = f(x + dx) - f(x)$
- Linear change: $dy = f'(x)\,dx$

Example 1: Use differentials to estimate how $f(x) = \dfrac{1}{x^2}$ changes as x increases from 2 to 2.1. Compare your estimate to the actual change.

Solution: Write $f(x) = x^{-2}$. By the power rule, $f'(x) = -2x^{-3} = -\dfrac{2}{x^3}$.

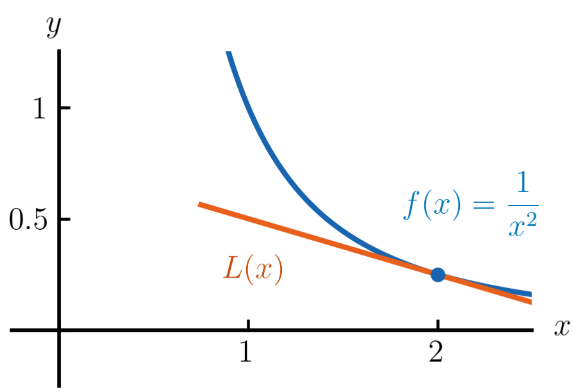
The change in y is $\Delta y \approx dy = f'(x)\,dx$. The distance from 2 to 2.1 is $\Delta x = 2.1 - 2 = 0.1$, so $dx = \Delta x = 0.1$.

$$dy = f'(2)\,dx = -\frac{2}{2^3}\,(0.1) = -0.025.$$

The change in y is approximately $dy = -0.025$. The actual change is

$$\Delta y = f(2.1) - f(2) = \frac{1}{(2.1)^2} - \frac{1}{2^2} \approx -0.02324.$$

The differential estimate is close to the actual change.



Example 2: A sphere has radius 10 cm and is painted with a coat of paint 2 mm thick. Use differentials to estimate the volume of paint used.

Solution: The volume of a sphere of radius r is $V(r) = \dfrac{4}{3}\pi r^3$. By the power rule, $V'(r) = 4\pi r^2$.

The change in volume is approximately $dV = V'(r)\,dr = 4\pi r^2\,dr$. The radius $r = 10$ cm. The paint is 2 mm thick which is 0.2 cm, so $dr = 0.2$ cm. Then

$$dV = 4\pi(10)^2(0.2) = 80\pi \text{ cm}^3 \approx 251.3 \text{ cm}^3.$$