Differentiability

We will show that if f is differentiable at a point, then f is continuous at that point. Then we will explore three types of points where functions are not differentiable.

Theorem **Differentiability Implies Continuity:** If f is differentiable at a, then f is continuous at a.

Proof: If $x \neq a$, then

$$f(x) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a).$$

So:

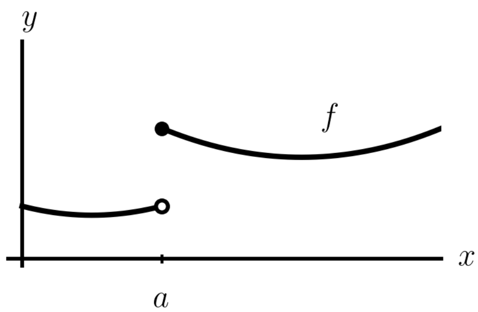
$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) + \lim_{x \to a} f(a) \\ &= f'(a) \cdot 0 + f(a) \\ &= f(a) \end{aligned}$$

Since $\lim_{x \to a} f(x) = f(a)$, f is continuous at a. $\square$

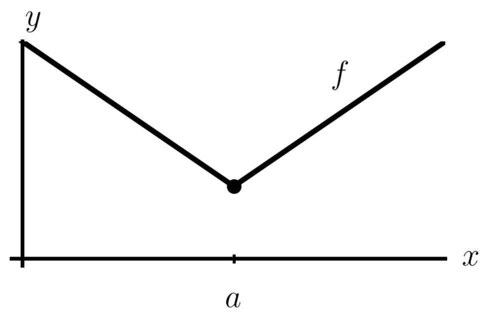
The contrapositive is also true: If f is not continuous at a, then f is not differentiable at a.

Careful: If f is continuous at a, then f may or may not be differentiable at a.

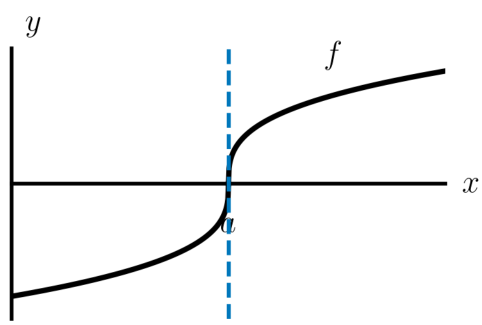
3 Types of Non-Differentiability: f is not differentiable at a if:



f is not continuous at a

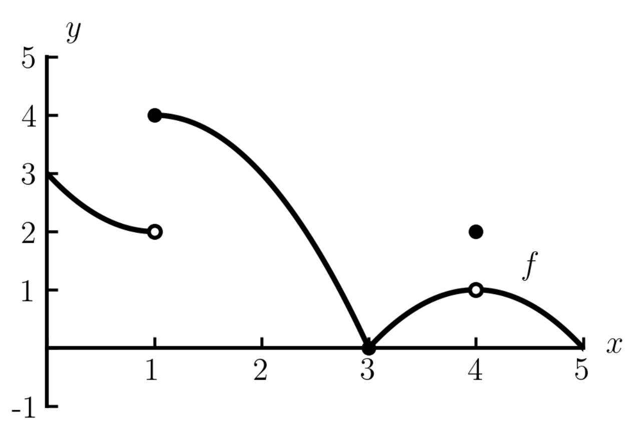


f has a sharp corner at a



f has a vertical tangent line at a

Example 1: For the function f in the graph, identify where f is not continuous and where f is not differentiable.



Solution: f is not continuous at $x = 1$ (jump discontinuity) and $x = 4$ (removable discontinuity).

f is not differentiable at $x = 1$ and $x = 4$ (discontinuities), and $x = 3$ (sharp corner).