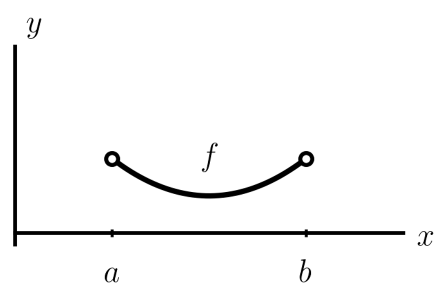
Continuous Functions

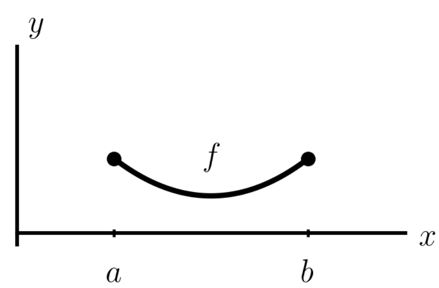
In this lesson, we will explore continuity on an interval, use the properties of continuous functions, and apply the Intermediate Value Theorem.

Many functions we have seen are continuous at every point in their domain, so we will start exploring continuity on intervals.

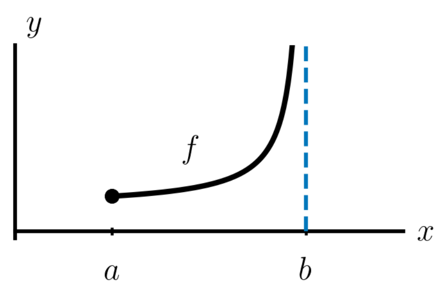
Definition **Continuity on an Interval:** A function f is continuous on an interval if it is continuous at every point in the interval. If the interval contains its endpoints, then f must be continuous from the left or right at the endpoints.



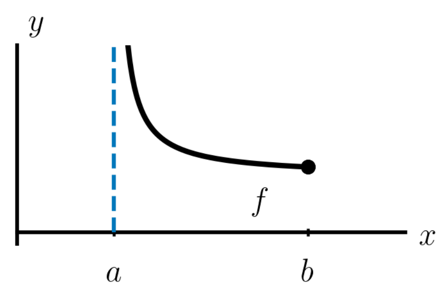
Continuous on (a, b)



Continuous on $[a, b]$



Continuous on $[a, b)$



Continuous on $(a, b]$

Theorem Continuity of Combinations of Continuous Functions: If f and g are continuous at a, and c is a constant, then each of these functions are also continuous at a:

- cf
- $f + g$
- $f - g$
- fg
- $\frac{f}{g}$ if $g(a) \neq 0$

Previously we explored this Direct Substitution Property: if f is a polynomial or rational function, and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$. This is the definition of continuity at a, so this property holds for all continuous functions.

Theorem Direct Substitution Property: If f is continuous at a, then we may use direct substitution to evaluate $\lim_{x \to a} f(x) = f(a)$.

Theorem Continuity of Common Functions: The following functions are continuous on their domains:

- Polynomials
- Rational functions
- Root functions
- Exponential and logarithmic functions
- Trigonometric and inverse trigonometric functions

Careful: If a is not in the domain of f, then f is not continuous at a.

Example 1: Where is $f(x) = \frac{(x-4)(x+2)}{(x-4)(x+3)}$ continuous?

Solution:

f is undefined when $x = 4$ or $x = -3$. Since f is a rational function, it is continuous on its domain. So, f is continuous for all x except $x = -3$ and $x = 4$.

Example 2: Find $\lim_{x \to \pi} \dfrac{3\cos x}{x + \sin x}$.

Solution:

Since x, $\sin x$, and $\cos x$ are all continuous functions, and the denominator $x + \sin x$ is nonzero at $x = \pi$, the function $\dfrac{3\cos x}{x + \sin x}$ is continuous at $x = \pi$.
Therefore, we can use direct substitution:

$$\lim_{x \to \pi} \frac{3\cos x}{x + \sin x} = \frac{3\cos \pi}{\pi + \sin \pi} = \frac{3(-1)}{\pi + 0} = -\frac{3}{\pi}$$

Theorem Continuity of Composite Functions: If g is continuous at a, and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Example 3: Where is $f(x) = \sin(3x^2 - 4)$ continuous?

Solution:

Since $\sin x$ and $3x^2 - 4$ are both continuous for all x, by the continuity of composite functions, $f(x) = \sin(3x^2 - 4)$ is continuous for all x.

Theorem Limit of a Composite Function: If f is continuous at b, and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$.

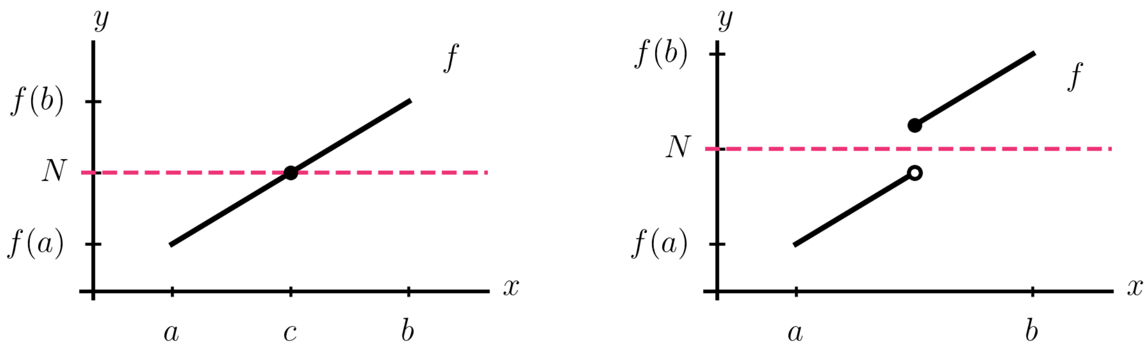
Example 4: Find $\lim_{x \to 0} \sin\left(\dfrac{\pi x}{x(x+2)}\right)$.

Solution:

Since $\sin x$ is continuous for all x, we can move the limit inside:

$$\begin{aligned}\lim_{x \to 0} \sin\left(\frac{\pi x}{x(x+2)}\right) &= \sin\left(\lim_{x \to 0} \frac{\pi x}{x(x+2)}\right) \\ &= \sin\left(\lim_{x \to 0} \frac{\pi}{x+2}\right) = \sin\left(\frac{\pi}{2}\right) = 1\end{aligned}$$

Theorem Intermediate Value Theorem: Let f be continuous on the closed interval $[a, b]$, and let N be any number strictly between $f(a)$ and $f(b)$. Then there is at least one point c in (a, b) such that $f(c) = N$.



Example 5: Show that the equation $x^5 = -2x + 4$ has a solution in the interval $[1, 2]$.

Solution:

Rearrange to get: $x^5 + 2x - 4 = 0$. Let $f(x) = x^5 + 2x - 4$. f is continuous on $[1, 2]$.

$f(1) = 1^5 + 2(1) - 4 = -1$ and $f(2) = 2^5 + 2(2) - 4 = 32$.

Since 0 is strictly between $f(1)$ and $f(2)$, by the Intermediate Value Theorem, there exists at least one point c in $(1, 2)$ such that $f(c) = 0$, which solves the equation.