

Average Value of a Function

In this lesson, we will define the average value of a function, interpret it geometrically, and apply the Mean Value Theorem for Integrals.

Average Value of a Function

The average of the numbers $y_1, \dots, y_n$ is

$$\frac{y_1 + \dots + y_n}{n} = \frac{1}{n} \sum_{i=1}^n y_i.$$

For a function, partition $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$ and pick sample points x_i^* . The average of the sampled function values is

$$\frac{1}{n} \sum_{i=1}^n f(x_i^*).$$

Since $n = \frac{b-a}{\Delta x}$, we get

$$\frac{1}{n} \sum_{i=1}^n f(x_i^*) = \frac{1}{\frac{b-a}{\Delta x}} \sum_{i=1}^n f(x_i^*) = \frac{1}{b-a} \sum_{i=1}^n f(x_i^*) \Delta x.$$

As n increases, the Riemann sum approaches the definite integral. So,

$$\bar{f} = \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \frac{1}{b-a} \int_a^b f(x) dx.$$

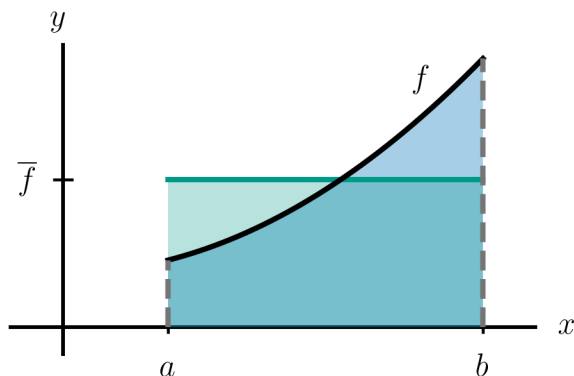
Definition Average Value of a Function: If $a < b$ and f is integrable on $[a, b]$, the **average value** of f on $[a, b]$ is

$$\boxed{\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx}.$$

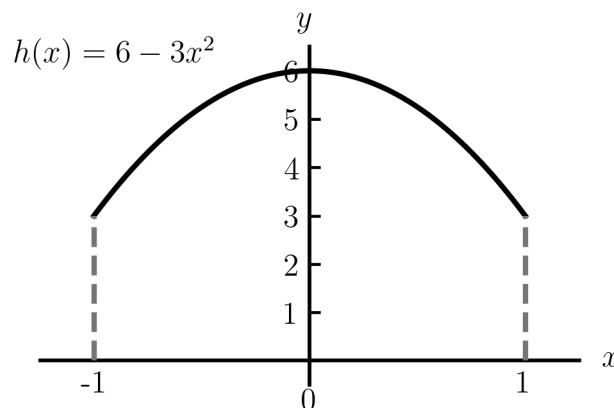
The notation f_{avg} is also commonly used for the average value.

Geometric Interpretation: Moving $(b - a)$ to the other side gives $\underbrace{(b - a)}_{\text{width}} \underbrace{\bar{f}}_{\text{height}} = \underbrace{\int_a^b f(x) dx}_{\text{area under curve}}.$

For a positive function, the rectangle with width $b - a$ and height $\bar{f}$ has the same area as the region under f on $[a, b]$.



Example 1: The height of a curved arch is $h(x) = 6 - 3x^2$ meters for $-1 \leq x \leq 1$, where x is horizontal distance in meters from the center. Find the average height of the arch.



Solution: We use the average-value formula. An antiderivative is $H(x) = 6x - x^3$. By FTC2,

$$\begin{aligned} \bar{h} &= \frac{1}{1 - (-1)} \int_{-1}^1 (6 - 3x^2) dx \\ &= \frac{1}{2} [6x - x^3]_{-1}^1 \\ &= \frac{1}{2} (5 - (-5)) \\ &= 5 \end{aligned}$$

So, the average height of the arch is $\boxed{\bar{h} = 5 \text{ m}}.$

Mean Value Theorem for Integrals

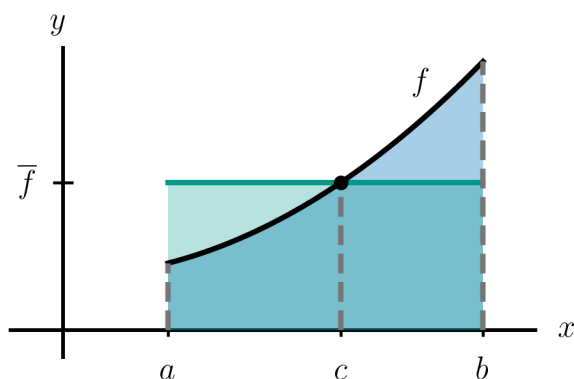
Mean Value Theorem: If G is continuous on $[a, b]$ and differentiable on (a, b) , then there is at least one point c in (a, b) such that $G'(c) = \frac{G(b) - G(a)}{b - a}$.

Let's apply the MVT to an area function $F(x) = \int_a^x f(t) dt$.

By the MVT, there is at least one point c in (a, b) such that $F'(c) = \frac{F(b) - F(a)}{b - a}$.

By FTC1, $F'(c) = f(c)$. Also, $F(b) - F(a) = \int_a^b f(t) dt$. So,

$$f(c) = \frac{1}{b - a} \int_a^b f(t) dt = \bar{f}.$$



Theorem Mean Value Theorem for Integrals: If $a < b$ and f is continuous on $[a, b]$, then there is at least one point c in (a, b) such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x) dx = \bar{f}.$$

Equivalently,

$$\int_a^b f(x) dx = f(c)(b - a).$$

Example 2: The arch from Example 1 has height $h(x) = 6 - 3x^2$. Find every c in $(-1, 1)$ where the arch has its average height, guaranteed by the MVT for Integrals.

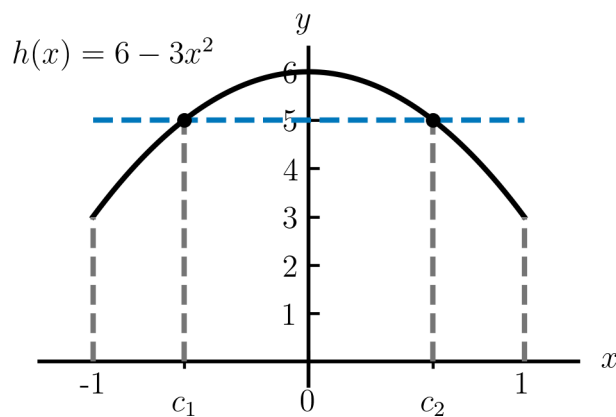
Solution: From Example 1, the average height is $\bar{h} = 5$. We solve $h(c) = \bar{h}$:

$$\begin{aligned}6 - 3c^2 &= 5 \\3c^2 &= 1 \\c^2 &= \frac{1}{3}.\end{aligned}$$

So,

$$c = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3} \approx \pm 0.577.$$

Both values are in $(-1, 1)$, and $h(c) = 5 = \bar{h}$ at each one. The height equals the average height at both points.



Now try this interactive problem on the website to practice finding an average value and a point where a function reaches it.

Interactive Problem 1: The height of a curved ramp is $h(x) = 1 + 3x^2$ feet for $1 \leq x \leq 4$. Find its average height and a point where the ramp has that height.

- (a) The expression that gives the average height of the ramp is

$$\bar{h} = \frac{1}{3} \int_1^4 (1 + 3x^2) dx.$$

- (b) An antiderivative $H(x)$ of $h(x) = 1 + 3x^2$ with constant term 0 is

$$H(x) = x + x^3.$$

- (c) Evaluating the average height gives

$$\bar{h} = 22.$$

- (d) The Mean Value Theorem for Integrals guarantees some c in $(1, 4)$ such that $h(c) = 22$.

- (e) Setting $h(c) = \bar{h}$ gives

$$1 + 3c^2 = 22$$

$$3c^2 = 21$$

$$c^2 = 7$$

$$c = \sqrt{7}.$$