

Area Functions

In this lesson, we will define area functions, compute an area function using geometry, and connect its derivative to the original function.

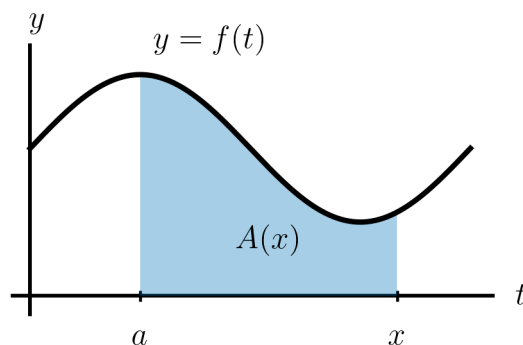
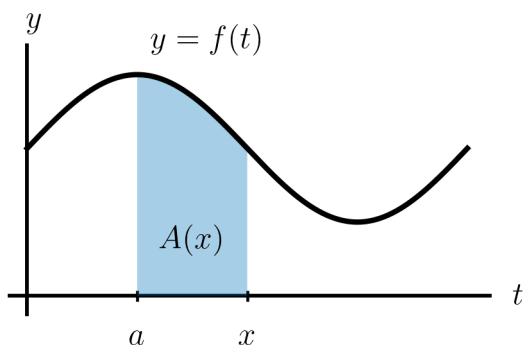
A definite integral $\int_a^b f(t) dt$ gives the net area between the graph of f and the horizontal axis from a to b . Replacing b with the variable x gives the accumulated net area as a function of x .

Definition Area Function: Let f be continuous on an interval containing a . The **area function measured from a** is the function

$$A(x) = \int_a^x f(t) dt$$

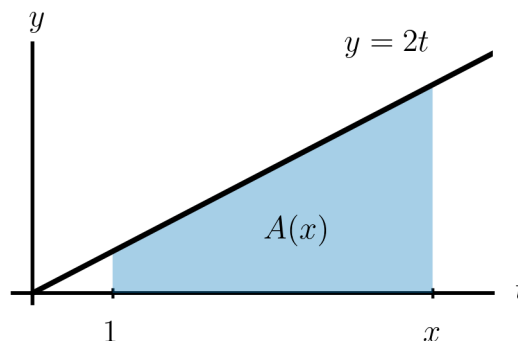
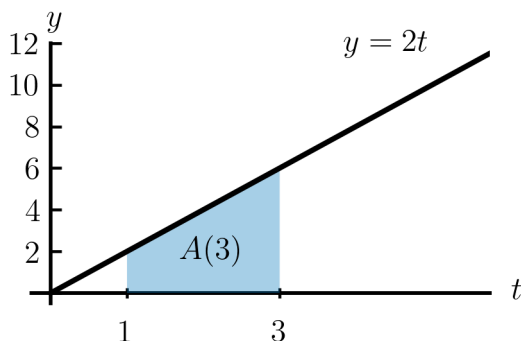
for all x in the interval.

Here x is the input and upper limit of integration. Since x is already used, we use t as the variable of integration. $A(x)$ gives the net area between the graph of f and the horizontal axis from $t = a$ to $t = x$.



As x moves to the right, more area under the curve is accumulated and $A(x)$ changes.

Example 1: Let $f(t) = 2t$, and let $A(x)$ be the area function measured from $a = 1$. Find $A(3)$, a formula for $A(x)$ for $x \geq 1$, and the derivative $A'(x)$.



Solution: For $A(3)$, the region under f from $t = 1$ to $t = 3$ is a trapezoid. The area of a trapezoid is given by $A = \frac{1}{2}(b_1 + b_2)h$. The bases (vertical sides) are $f(1) = 2$ and $f(3) = 6$, and the width is $3 - 1 = 2$. So,

$$A(3) = \frac{1}{2}(2 + 6)(2) = \boxed{8}$$

For $x \geq 1$, the trapezoid has bases (vertical sides) $f(1) = 2$ and $f(x) = 2x$, with width $x - 1$. So,

$$\begin{aligned} A(x) &= \frac{1}{2}(2 + 2x)(x - 1) \\ &= (x + 1)(x - 1) \\ &= x^2 - 1 \end{aligned}$$

So, in integral form,

$$A(x) = \int_1^x 2t \, dt = \boxed{x^2 - 1}$$

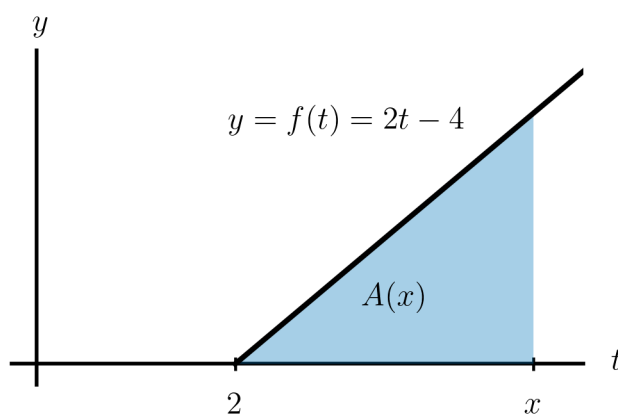
Now differentiate the formula for the area function:

$$A'(x) = \frac{d}{dx}(x^2 - 1) = \boxed{2x} = \boxed{f(x)}$$

The derivative of the area function is the original function! This is not a coincidence. In the next lesson, the Fundamental Theorem of Calculus will explain why this happens.

Now try this interactive problem on the website to practice finding an area function and checking its derivative.

Interactive Problem 1: Let $f(t) = 2t - 4$ and consider the area function $A(x) = \int_2^x (2t - 4) dt$ measured from $a = 2$, for $x \geq 2$. We will find a formula for $A(x)$, then check the derivative $A'(x)$.



(a) The area function at the starting point is $A(2) = 0$.

(b) For $x \geq 2$, the triangular region has base length $\text{base} = x - 2$.

(c) The height of the triangle at $t = x$ is $\text{height} = 2x - 4$.

(d) Using the area formula for a triangle $A = \frac{1}{2}bh$,

$$\begin{aligned} A(x) &= \frac{1}{2}(x - 2)(2x - 4) \\ &= (x - 2)^2. \end{aligned}$$

(e) Differentiating the formula $A(x) = (x - 2)^2$,

$$A'(x) = 2(x - 2) = 2x - 4 = f(x).$$

So, $A(x) = (x - 2)^2$ and $A'(x) = 2x - 4 = f(x)$. The area function is the antiderivative of the original function.